ANDERSON JUNIOR COLLEGE
2013 Preliminary Examinations
H2 MATHEMATICS (JC2)
PAPER 1 (Solutions)

1

For small θ, $\cos\theta \approx 1-\frac{1}{2}\theta^2$

$$y=\frac{1}{5+2\cos^2(ax)}$$
$$=\frac{1}{5+1+\cos(2ax)}$$
$$\simeq \frac{1}{5+1+1-2a^2x^2}$$
$$=\frac{1}{7-2a^2x^2}$$

or

$$y=\frac{1}{5+2\cos^2(ax)}$$
$$\simeq \frac{1}{5+2\left(1-\frac{a^2x^2}{2}\right)^2}$$
$$\simeq \frac{1}{5+2\left(1-a^2x^2\right)}$$
$$=\frac{1}{7-2a^2x^2}$$

$$y=\frac{1}{(7)\left(1-\frac{2a^2}{7}x^2\right)}=\frac{1}{7}\left(1-\frac{2a^2}{7}x^2\right)^{-1}\simeq \frac{1}{7}\left(1+\frac{2a^2}{7}x^2\right)$$
$$=\frac{1}{7}+\frac{2a^2}{49}x^2$$
$$\equiv \frac{1}{7}+\frac{18}{49}x^2$$

$\therefore\ 2a^2=18 \quad \Rightarrow \quad a=3$

3

$2u=\cos x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{\sin x}{2}$

$$\int_0^{\frac{1}{2}} 4u^2\sqrt{1-4u^2}\,\mathrm{d}u$$
$$=\int_{\frac{\pi}{2}}^{0}\cos^2 x\sqrt{1-\cos^2 x}\left(-\frac{\sin x}{2}\right)\mathrm{d}x$$
$$=-\frac{1}{2}\int_{\frac{\pi}{2}}^{0}\cos^2 x\sin^2 x\,\mathrm{d}x$$
$$=\frac{1}{8}\int_0^{\frac{\pi}{2}}\sin^2 2x\,\mathrm{d}x$$
$$=\frac{1}{16}\int_0^{\frac{\pi}{2}}1-\cos 4x\,\mathrm{d}x$$

$$= \frac{1}{16}\left[x - \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{32}$$

$$\int_0^{\frac{1}{2}} \left(\sqrt{1-4u^2} \right)^3 \mathrm{d}u$$

$$= \left[u\left(1-4u^2\right)^{\frac{3}{2}} \right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} u\left(\frac{3}{2}\left(1-4u^2\right)^{\frac{1}{2}}\left(-8u\right) \right)\mathrm{d}u$$

$$= 3\int_0^{\frac{1}{2}} 4u^2\left(1-4u^2\right)^{\frac{1}{2}}\,\mathrm{d}u$$

$$= 3\left(\frac{\pi}{32} \right)$$

$$= \frac{3\pi}{32}$$

Alternatively,

$$\int_0^{\frac{1}{2}} \left(1-4u^2\right)\left(\sqrt{1-4u^2} \right)\mathrm{d}u$$

$$= \int_0^{\frac{1}{2}} \sqrt{1-4u^2}\,\mathrm{d}u - \int_0^{\frac{1}{2}} \left(4u^2\right)\left(\sqrt{1-4u^2} \right)\mathrm{d}u$$

$$= -\frac{1}{2}\int_{\frac{\pi}{2}}^{0} \sin^2 x\,\mathrm{d}x - \frac{\pi}{32}$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 1-\cos 2x\,\mathrm{d}x - \frac{\pi}{32}$$

$$= \frac{1}{4}\left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} - \frac{\pi}{32}$$

$$= \frac{3\pi}{32}$$

3 (i) From the tables, $\mathrm{ff}(6) = \mathrm{f}(14) = 26$

$\mathrm{f}^{-1}(8) = 4$

(ii) Since $R_g = [0, 1] \subseteq D_f = [0, \infty)$, fg exists.

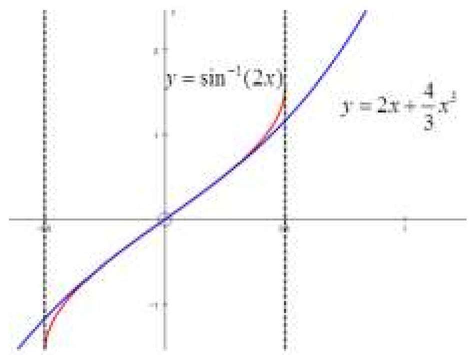
(iii) $D_{fg} = D_g = \left[0, \frac{\pi}{8} \right]$.

Taking R_g as new D_f, $R_{fg} = [-2, -0.5]$ (f is increasing)

	$\lvert \mathrm{fg}(x) \rvert < 1 \Rightarrow \lvert \mathrm{f}(\tan 2x) \rvert < 1$ But $R_{fg} = [-2, -0.5]$, $-1 < \mathrm{f}(\tan 2x) \le -0.5$ $\Rightarrow \frac{1}{\sqrt{3}} < \tan 2x \le 1$ $\Rightarrow \frac{\pi}{6} < 2x \le \frac{\pi}{4} \Rightarrow \frac{\pi}{12} < x \le \frac{\pi}{8}$
4	$y = \sin^{-1}(2x)$ diff. w.r.t. x $\frac{dy}{dx} = \frac{2}{\sqrt{1-(2x)^2}}$ $\sqrt{1-4x^2}\frac{dy}{dx} = 2$ $(1-4x^2)\left(\frac{dy}{dx}\right)^2 = 4$ (i) $2(1-4x^2)\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + (-8x)\left(\frac{dy}{dx}\right)^2 = 0$ $\left(\frac{dy}{dx}\right)\left[2(1-4x^2)\left(\frac{d^2y}{dx^2}\right) - 8x\left(\frac{dy}{dx}\right)\right] = 0$ Since $\frac{dy}{dx} \neq 0$, $2(1-4x^2)\left(\frac{d^2y}{dx^2}\right) - 8x\left(\frac{dy}{dx}\right) = 0$ $2(1-4x^2)\left(\frac{d^3y}{dx^3}\right) + (-16x)\left(\frac{d^2y}{dx^2}\right) - 8\left(\frac{dy}{dx}\right) - 8x\left(\frac{d^2y}{dx^2}\right) = 0$ Sub $x = 0$ $y = \sin^{-1} 0 = 0$, $\frac{dy}{dx} = \frac{2}{\sqrt{1-0^2}} = 2$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 8$ $\mathrm{f}(0) = 0 \quad \mathrm{f}'(0) = 2 \quad \mathrm{f}''(0) = 0 \quad \mathrm{f}'''(0) = 8$ $\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \frac{\mathrm{f}''(0)}{2!}x^2 + \frac{\mathrm{f}'''(0)}{3!}x^3 +$ $\mathrm{f}(x) = 2x + \frac{4}{3}x^3 + \ldots$

(ii)

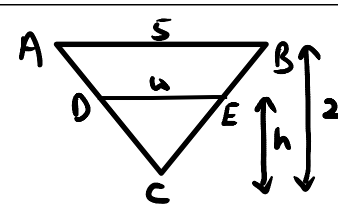
$$\text{Volume} = \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(2x + \frac{4}{3}x^3 \right)^2 \mathrm{d}x$$



The approximated volume is an under-estimation of the actual volume. This can be seen from the above diagram, the region under the graph of $y = \sin^{-1}(2x)$ is larger than the region under the graph of $y = 2x + \frac{4}{3}x^3$.

5 (i) Using similar triangles ABC and DEC,

$$\frac{w}{5} = \frac{h}{2} \Rightarrow w = \frac{5}{2}h$$



Volume of the water in the tank = Base area $\times$ length

$$V = \frac{1}{2}wh \times 8$$

$$V = 4hw = 4h \times \frac{5}{2}h = 10\,h^2$$

(ii) $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \dfrac{\mathrm{d}V}{\mathrm{d}h}\dfrac{\mathrm{d}h}{\mathrm{d}t}$

$\dfrac{\mathrm{d}V}{\mathrm{d}t} = 20h\dfrac{\mathrm{d}h}{\mathrm{d}t}$ ---- (1)

At $t = 2$ seconds, $V = 2(5) = 10$ m^3

To find h, $10 = 10\,h^2$

$h = 1$ m

	$20\frac{\mathrm{d}h}{\mathrm{d}t}=5$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{4} \quad m/s$ (iii) from (1), $2h=20h\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{10} \quad \mathrm{m/s}$ Since the rate of change for h is a constant, time taken for h from 1 to 2 m = 10 seconds Therefore the time taken for the trough to be completely filled is 12 seconds.
6 (a)	First term =1, common difference = d S_5, S_{10}, S_{20} form a GP $\Rightarrow \frac{S_{10}}{S_5}=\frac{S_{20}}{S_{10}}$ $\Rightarrow \left(\frac{10}{2}(2+9d)\right)^2=\frac{5}{2}(2+4d).\frac{20}{2}(2+19d)$ $\Rightarrow (2+9d)^2=(2+4d).(2+19d)$ $\Rightarrow 5d^2-10d=0$ $\Rightarrow d=2$ or $d=0$ (rejected as AP is increasing) $\frac{2S_n}{S_{n+1}-100}>1$ $\frac{2n^2}{(n+1)^2-100}>1$ $\frac{2n^2-(n+1)^2+100}{(n+1)^2-100}>0$ $\frac{n^2-2n+99}{(n+1)^2-100}>0$ Since $n^2-2n+99>0$ as discriminant < 0 and coefficient of n^2 is +ve. $(n+1)^2-100>0$ $(n+1+10)(n+1-10)>0$ $(n+11)(n-9)>0$ $n>9$ $\therefore$ least n is 10.

6 (b)

nth month	Outstanding amt owed at the start of the month (in hundreds)	Outstanding amt owed at the end of the nth month (in hundreds)
1	34	34
2	(34)2	(34)2
3	$(34)2^2$	$34(2)^2$-70
4	$2\left(34(2)^2-70\right)$	$34(2)^3-70(2)-70$
5	$2\left(34(2)^3-70(2)-70\right)$	$34(2)^4-70(2)^2-70(2)-70$
n		$34(2)^{n-1}-70(2)^{n-3}-..........-70(2)-70$

Total amount of money owed at the end of nth month

$$= 100(34(2)^{n-1}-70(2)^{n-3}-..........-70(2)-70)$$

$$=100\left(34(2)^{n-1}-\frac{70((2)^{n-2}-1)}{2-1}\right)$$

$$=100\left(70-2^{n-1}\right)$$

To be free from debt,

$$70-2^{n-1}\leq 0$$

$$\Rightarrow 2^n \geq 140$$

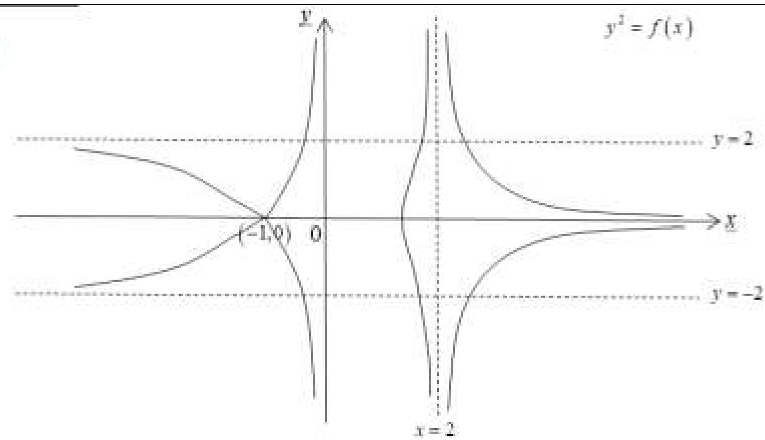
$$\Rightarrow n \geq \frac{\ln 140}{\ln 2}=7.129$$

Least n = 8

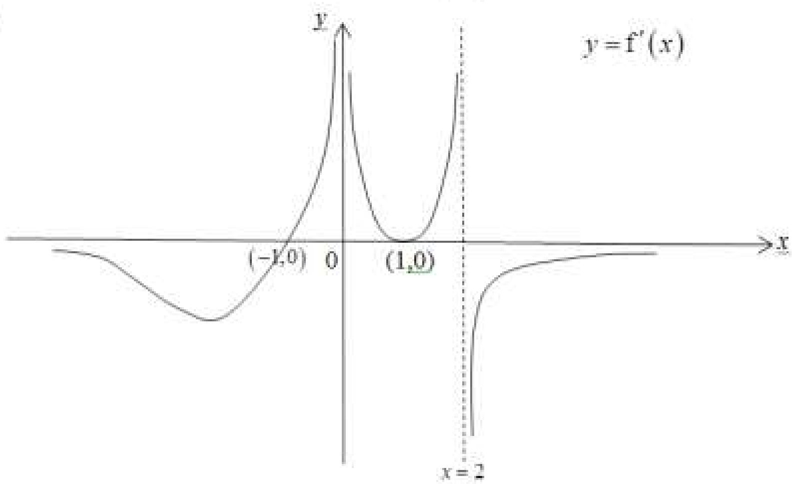
Earliest month to be free from debt = Jan 2014

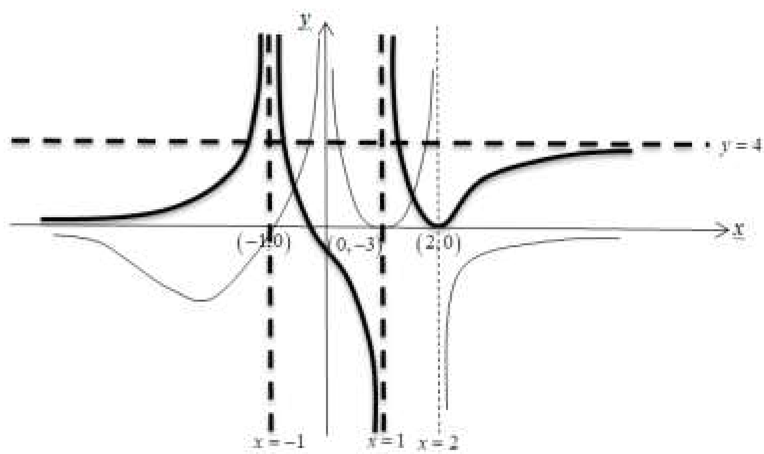
7

i)



ii)





M1 – shape

M1 – labeling & asymptotes

From graph, there are 2 points of intersection. Thus, there are 2 solutions.

8

(i) The points $Q(0,5,0)$ is on plane p

$$\Rightarrow \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} a \\ \\ 4 \end{pmatrix} \quad \Rightarrow \quad 5b = 15 \quad \Rightarrow \quad b = 3$$

(ii) $$\sin\alpha = \frac{\left| \begin{pmatrix} a \\ 3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} \right|}{\left| \begin{pmatrix} a \\ 3 \\ 4 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right|} \qquad \textbf{i.e.} \quad \sin\alpha = \frac{|a|}{\sqrt{a^2 + 25}}$$

(iii) The angle between $\begin{pmatrix} a \\ b \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is obtuse $\Rightarrow \begin{pmatrix} a \\ b \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} \quad \Rightarrow \quad a < 0$

$$\alpha = 45° \Rightarrow \frac{1}{\sqrt{2}} = \frac{|a|}{\sqrt{a^2 + 25}}$$

$$\Rightarrow a^2 + 25 = 2|a|^2$$

Since $|a|^2 = a^2$,

$$a^2 = 25$$

Since $a < 0$, $a = -5$

(iv) Equation of plane p : $-5x + 3y + 4z = 15$

Equation of x-z plane : $y = 0$

Equation of x-y plane : $z = 0$

Solving simultaneously

$$x = -3 \;,\; y = 0 \;,\; z = 0 \qquad \text{i.e.} \quad \overrightarrow{OW} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

(v) M $(1,0,5)$ is on plane p (given) and

is also on the x-z plane (y-coordinates of $M = 0$)

$\Rightarrow M$ is on their line of intersection l.

Points M and W are on l , and the shortest distance of Q to the line l

	$= \frac{\lvert\overrightarrow{WQ} \times \overrightarrow{WM}\rvert}{\lvert\overrightarrow{WM}\rvert}$ $\frac{\left\lvert\begin{pmatrix}3\\5\\0\end{pmatrix}\times\begin{pmatrix}4\\0\\5\end{pmatrix}\right\rvert}{\sqrt{4^2+5^2}} = \frac{\left\lvert\begin{pmatrix}25\\-15\\-20\end{pmatrix}\right\rvert}{\sqrt{4^2+5^2}} = \sqrt{\frac{1250}{41}}$ units or $\frac{\lvert\overrightarrow{MQ} \times \overrightarrow{WM}\rvert}{\lvert\overrightarrow{WM}\rvert}$
9 (a)	(i) Let P_n be the proposition that $u_n = \frac{a(n+1)n}{(n-1)!}$ for $n \in \mathbb{Z}$ For n = 1, LHS of P_1 = $2a$ RHS of $P_1 = = \frac{a(1+1)1}{(1-1)!} = \frac{2a}{1} = 2a$ = LHS of P_1 Thus, P_1 is true. Assume that P_k is true for some $k \in \mathbb{Z}$. i.e. $u_k = \frac{a(k+1)k}{(k-1)!}$ To prove that P_{k+1} is true: $u_{k+1} = \frac{k+2}{k^2}u_k$ $= \frac{(k+2)}{k^2}\frac{a(k+1)k}{(k-1)!}$ $= \frac{a(k+2)(k+1)}{k(k-1)!}$ $= \frac{a(k+2)(k+1)}{k!}$ Thus P_{k+1} is true. Since P_1 is true, and P_k true $\Rightarrow P_{k+1}$ is true. By Mathematical Induction, P_n is true for all $n \in \mathbb{Z}$ (ii) Consider

	$u_{n+1}-u_n=\frac{n+2}{n^2}u_n-u_n$ $=u_n(\frac{n+2}{n^2}-1)$ $=u_n(\frac{n+2-n^2}{n^2})$ $=-u_n\frac{(n+1)(n-2)}{n^2}<0$ since for $n\geq 3$ and $a>0$, $(n-2)>0$ and $u_n=\frac{a(n+1)(n)}{(n-1)!}>0$ $\therefore u_{n+1}<u_n$ for $n\geq 3$ Alternatively, $u_{n+1}-u_n$ $=\frac{a(n+2)(n+1)}{n!}-\frac{a(n+1)(n)}{(n-1)!}$ $=\frac{a(n+1)}{n!}\left(n+2-n^2\right)$ $=-\frac{a(n+1)^2(n-2)}{n!}$ Since $n\geq 3$ and $a\geq 0$, $(n+1)^2>0$ and $(n-2)>0$ $\therefore u_{n+1}-u_n<0$ for $n\geq 3$ $\sum_{r=3}^{16}u_r$ $=u_3+u_4+u_5+...+u_{16}$ $<u_3+u_3+u_3+...+u_3$ since $u_{r+1}<u_r$ for $r\geq 3$ $=14u_3$ $=84a$ since $u_3=6a$ $\therefore \sum_{r=3}^{16}u_r<84a$
9 (b)	

	$\sum_{r=2}^{n}\left(\int_2^r \frac{2}{x^2-1}\,\mathrm{d}x\right)$ $=\sum_{r=2}^{n}\left(\left[\ln\frac{x-1}{x+1}\right]_2^r\right)$ $=\sum_{r=2}^{n}\left(\ln\frac{r-1}{r+1}+\ln 3\right)$ $=\sum_{r=2}^{n}\left(\ln\frac{r-1}{r+1}\right)+\sum_{r=2}^{n}(\ln 3)$ $=\sum_{r=2}^{n}\left[(\ln r-1)-(\ln r+1)\right]+(n-1)\ln 3$ $=\begin{pmatrix} \ln(1) & - & \ln(3) \\ \ln(2) & - & \ln(4) \\ \ln(3) & - & \ln(5) \\ \vdots & & \vdots \\ \ln(n-2) & - & \ln(n) \\ \ln(n-1) & - & \ln(n+1) \end{pmatrix}+(n-1)\ln(3)$ $=\ln 2-\ln(n)-\ln(n+1)+\ln 3^{n-1}$ $=\ln\frac{2(3)^{n-1}}{n(n+1)}$ $\sum_{r=6}^{12}\left(\int_2^{r+2} \frac{2}{x^2-1}\,\mathrm{d}x\right)$ $=\sum_{m-2=6}^{m-2=12}\left(\int_2^m \frac{2}{x^2-1}\,\mathrm{d}x\right)$ $=\sum_{m=8}^{14}\left(\int_2^m \frac{2}{x^2-1}\,\mathrm{d}x\right)$ $=\sum_{m=2}^{14}\left(\int_2^m \frac{2}{x^2-1}\,\mathrm{d}x\right)-\sum_{m=2}^{7}\left(\int_2^m \frac{2}{x^2-1}\,\mathrm{d}x\right)$ $=\ln\frac{2(3)^{13}}{14(15)}-\ln\frac{2(3)^6}{7(8)}$ $=\ln\frac{4(3)^6}{5}$
10 (a)	(i)

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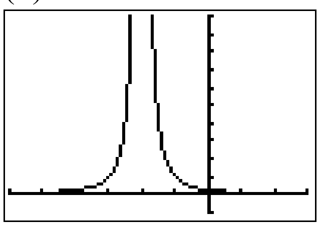
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x+4}{\left(x^2+4x+4\right)^2}$$

$$= -\frac{2(x+2)}{(x+2)^4}$$

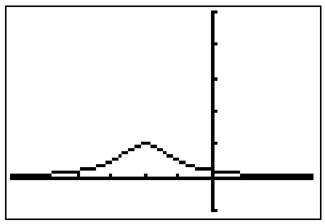
$$= -\frac{2}{(x+2)^3}$$

Since $\frac{2}{(x+2)^3} \neq 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$, there are no stationary point when $C = 4$.

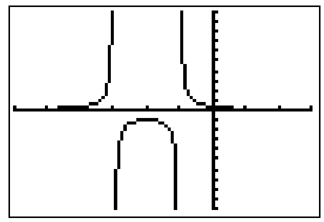
(ii)



When *C* = 4



When *C* > 4



When *C* < 4

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<td>10 (b)</td>
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Let w be the amount of water present in the leaf at any time t.

$\frac{\mathrm{d}r}{\mathrm{d}t} = m\frac{\mathrm{d}w}{\mathrm{d}t}$ where m is a positive constant

Since $\frac{\mathrm{d}w}{\mathrm{d}t} = 8r - \frac{1}{\pi}\left(\pi r^2\right)$ where m is a positive constant.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = m\left(8r - r^2\right)$$

When $r = 2$, $\frac{\mathrm{d}r}{\mathrm{d}t} = 6$.

$$6 = m(16-4)$$

$$m = \frac{1}{2}$$

$$\therefore \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2}\left(8r - r^2\right)$$

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	$\int \frac{1}{r^2-8r}\,\mathrm{d}r = \int -\frac{1}{2}\,\mathrm{dt}$ $\int \frac{1}{(r-4)^2-4^2}\,\mathrm{d}r = -\frac{1}{2}t + C$ [M1] $\frac{1}{8}\ln\left(\frac{r-4-4}{r-4+4}\right) = -\frac{1}{2}t + C$ $\frac{1}{8}\ln\left\|\frac{r-8}{r}\right\| = -\frac{1}{2}t + C$ $\ln\left\|\frac{r-8}{r}\right\| = -4t + 8C$ $\frac{r-8}{r} = \pm e^{-4t+8C} = Be^{-4t}$ where $B = \pm e^{8C}$ $r = \frac{8}{1-Be^{-4t}}$ When $t = 0$, $r = 4$ $4 = \frac{8}{1-Be^{0}}$ $B = -1$ $r = \frac{8}{1+e^{-4t}}$ As $t \to \infty$, $e^{-4t} \to 0$, $r \to 8$ The radius of the circular shaped leaf will grow to a radius of 8 cm for large values of t.
11 (a)	$s - w = 6i \Rightarrow$ The real part of s and w are the same. Let $s = a + bi$ and $w = a + ci$ $\therefore b - c = 6$ --- (1) $(a+bi)(a+ci) = 10$ $a^2 - bc + a(b+c)i = 10$ $\therefore a^2 - bc = 10$ and $b + c = 0$ $a = \pm 1$, $b = 3$ and $c = -3$ Since $a > 0$, $\therefore s = 1 + 3i$, $w = 1 - 3i$ Alternatively,

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Subst $w=\dfrac{10}{s}$ into $s-w=6i$,
$s-\dfrac{10}{s}=6i$
$s^2-10=(6i)s$
$s^2-(6i)s-10=0$
$s=\dfrac{6i\pm\sqrt{-36-4(-10)}}{2}=\pm1+3i$
Since Re(s) > 0, $s=1+3i$ and $w=1-3i$

Let $u=is$ and $v=iw$ and we would arrive at the original pair of given equations.

$\therefore u=-3+i$ and $v=3+i$

Alternatively,
$s=iv,\quad w=iu$
$\therefore v=3-i$ and $u=-3-i$
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<tr><td>11 (b)</td><td>
$\left|-3+\sqrt{3}i\right|=\sqrt{12}$ and $\arg(-3+\sqrt{3}i)=\dfrac{5\pi}{6}$

$z^3=\sqrt{12}e^{\frac{5\pi}{6}i+2k\pi i}$
$\Rightarrow z=12^{\frac{1}{6}}e^{\frac{5\pi}{18}i+\frac{2k\pi i}{3}},\quad k=0,\pm1$
$\Rightarrow z_1=12^{\frac{1}{6}}e^{\frac{-7\pi}{18}i},\quad z_2=12^{\frac{1}{6}}e^{\frac{5\pi}{18}i}$ and $z_3=12^{\frac{1}{6}}e^{\frac{17\pi}{18}i}$
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▲ ▲ ▲ are congruent triangles with $\left|z_1\right|=\left|z_2\right|=\left|z_3\right|=12^{\frac{1}{6}}$ and
∡ ∡ ∡

Area of triangle $Z_1Z_2Z_3=3\times\dfrac{1}{2}(12)^{\frac{1}{6}}(12)^{\frac{1}{6}}\sin\left(\dfrac{2\pi}{3}\right)$
$=\dfrac{3\sqrt{3}}{4}(12)^{\frac{1}{3}}$
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Let $c=e^{i\theta}$
$\therefore cz_2=e^{i\theta}\times e^{\frac{5\pi i}{18}}=e^{\left(\frac{5\pi}{18}+\theta\right)i}$
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	Since cz_2 is a positive real number, $\arg(cz_2)=0$ $\frac{5\pi}{18}+\theta=0 \quad \Rightarrow \quad \theta=-\frac{5\pi}{18}$ $c=e^{\frac{-5\pi i}{18}}$ Simliarly, we can also consider cz_2 or cz_3 The corresponding values for c would be $e^{\frac{-17\pi i}{18}}$ and $e^{\frac{7\pi i}{18}}$ respectively. Any one of the above 3 values of c is acceptable.