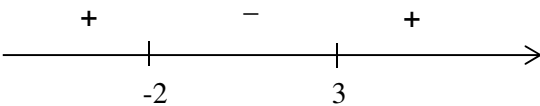
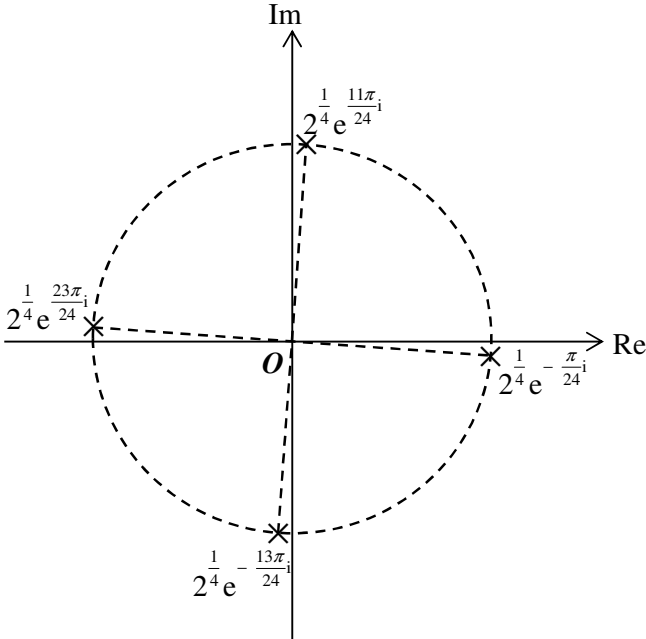


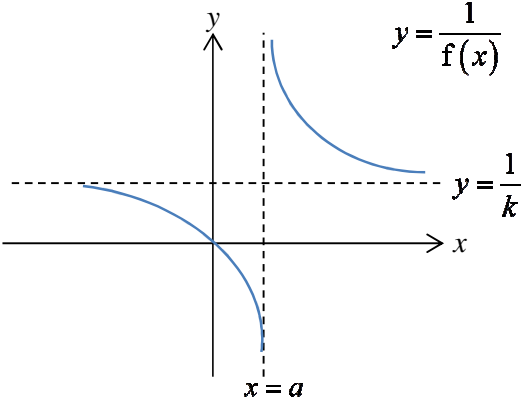
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Qn	Solution															
1	<p>Let x, y, z be the number of apples, oranges and pineapples respectively.</p> $x + y + z = 50$ $0.8x + 0.6y + 1.2z = 40$ $1.6x + 0.3y + 1.2z = 53$ <div style="display: flex; justify-content: space-around; margin: 10px 0;"> <div style="border: 1px solid black; padding: 5px; width: 45%;"> <p style="text-align: center; margin: 0;">SYSTEM MATRIX (3x4)</p> <table style="width: 100%; border-collapse: collapse; font-family: monospace;"> <tr><td style="border-right: 1px solid black;">[1</td><td style="border-right: 1px solid black;">1</td><td style="border-right: 1px solid black;">1</td><td style="border-right: 1px solid black;">50</td><td style="border-right: 1px solid black;">]</td></tr> <tr><td style="border-right: 1px solid black;">[.8</td><td style="border-right: 1px solid black;">.6</td><td style="border-right: 1px solid black;">1.2</td><td style="border-right: 1px solid black;">40</td><td style="border-right: 1px solid black;">]</td></tr> <tr><td style="border-right: 1px solid black;">[1.6</td><td style="border-right: 1px solid black;">.3</td><td style="border-right: 1px solid black;">1.2</td><td style="border-right: 1px solid black;">53</td><td style="border-right: 1px solid black;">]</td></tr> </table> <p style="margin-top: 10px;">(3,4)=53</p> <p style="font-size: small; margin: 0;">(MAIN)MODEVCLRLDADISOLVE)</p> </div> <div style="border: 1px solid black; padding: 5px; width: 45%;"> <p style="text-align: center; margin: 0;">SOLUTION</p> <p style="margin: 5px 0;">x1 = 23</p> <p style="margin: 5px 0;">x2 = 18</p> <p style="margin: 5px 0;">x3 = 9</p> <p style="font-size: small; margin: 0;">(MAIN)MODEVSYSHI STD IF 4+0)</p> </div> </div> <p>From GC, Adam bought 23 apples, 18 oranges and 9 pineapples.</p>	[1	1	1	50]	[.8	.6	1.2	40]	[1.6	.3	1.2	53]
[1	1	1	50]												
[.8	.6	1.2	40]												
[1.6	.3	1.2	53]												
2(i)	<p>Since x, y, z are the first three terms of a geometric progression,</p> $\frac{y}{x} = \frac{z}{y}$ $y^2 = xz$ $x = \frac{y^2}{z}$ <p>Since z, x, y are three consecutive terms of an arithmetic progression,</p> $x - z = y - x$ $2x = y + z$ <p>Solving the 2 above equations,</p> $2\left(\frac{y^2}{z}\right) = y + z$ $2y^2 = yz + z^2$ <p>Dividing throughout by y^2 :</p> $2 = \left(\frac{z}{y}\right) + \left(\frac{z^2}{y^2}\right)$ $\therefore \left(\frac{z}{y}\right)^2 + \left(\frac{z}{y}\right) - 2 = 0 \text{ (shown)}$															
(ii)	<p>Geometric progression has common ratio $r = \frac{z}{y}$.</p>															

	<p>Solving $\left(\frac{z}{y}\right)^2 + \left(\frac{z}{y}\right) - 2 = 0$ gives $r = 1$ or -2.</p> <p>Since $r \neq 1$, so sum to infinity of geometric progression does not exist.</p>
3(i)	$x^2 - 2x + 4 = (x^2 - 2x + 1^2) - 1^2 + 3$ $= (x-1)^2 + 3$ <p>Since $(x-1)^2 \geq 0$,</p> $(x-1)^2 + 3 > 0 \text{ for all real values of } x.$ <p>Since $x^2 - 2x + 4 > 0$ for all real values of x,</p> $\frac{(x^2 - 2x + 4)(x-3)}{(x+2)} \geq 0$ $\frac{(x-3)}{(x+2)} \geq 0$  <p>$x < -2$ or $x \geq 3$</p>
(ii)	<p>Using (ii), $\frac{(x^2 - 2 x + 4)(x - 3)}{(x + 2)} \geq 0$</p> $\Rightarrow x < -2 \text{ or } x \geq 3,$ <p>Since $x \geq 0$, reject $x < -2$</p> $ x \geq 3 \Rightarrow x \leq -3 \text{ or } x \geq 3$
4(i)	$z^4 = \sqrt{3} - i$ $ \sqrt{3} - i = 2$ $\arg(\sqrt{3} - i) = -\frac{\pi}{6}$

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	$z^4 = 2e^{-\frac{\pi}{6}i}$ $z^4 = 2e^{(2k\pi - \frac{\pi}{6})i}$ $z = 2^{\frac{1}{4}} e^{\frac{1}{4} \left(\frac{12k-1}{6} \right) \pi i}, \quad k = 0, \pm 1, 2$ $z = 2^{\frac{1}{4}} e^{-\frac{13\pi}{24}i}, \quad 2^{\frac{1}{4}} e^{-\frac{\pi}{24}i}, \quad 2^{\frac{1}{4}} e^{\frac{11\pi}{24}i}, \quad 2^{\frac{1}{4}} e^{\frac{23\pi}{24}i}$
(ii)	 <p>The cartesian equation is $x^2 + y^2 = \sqrt{2}$</p>
5(i)	<p>Let $\frac{5+x^2}{(2+x)(1-x)^2} = \frac{A}{2+x} + \frac{B}{(1-x)^2}$.</p> <p>Hence $5+x^2 = A(1-x)^2 + B(2+x)$.</p> <p>Let $x=1$, $5+1^2 = B(2+1)$ $\therefore B=2$</p> <p>Let $x=-2$, $5+2^2 = A(1+2)^2$ $\therefore A=1$</p> $\frac{5+x^2}{(2+x)(1-x)^2} = \frac{1}{2+x} + \frac{2}{(1-x)^2}$

(ii)	$\frac{5+x^2}{(2+x)(1-x)^2}$ $= \frac{1}{2+x} + \frac{2}{(1-x)^2}$ $= (2+x)^{-1} + 2(1-x)^{-2}$ $= 2^{-1} \left(1 + \frac{x}{2}\right)^{-1} + 2 \left[1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \dots\right]$ $= \frac{1}{2} \left(1 - \frac{x}{2} + \frac{x^2}{4} + \dots\right) + 2[1 + 2x + 3x^2 + \dots]$ $= \frac{5}{2} + \frac{15}{4}x + \frac{49}{8}x^2 + \dots$
(iii)	<p>For expansion of $(1-x)^{-2}$ to be valid, $-x < 1$ $x < 1$</p> <p>For expansion of $\left(1 + \frac{x}{2}\right)^{-1}$ to be valid, $\left \frac{x}{2}\right < 1$ $x < 2$</p> <p>Hence for the expansion of $\frac{5+x^2}{(2+x)(1-x)^2}$ to be valid, $x < 1,$ $-1 < x < 1.$</p>
6 (a) (i)	 <p>The graph shows a coordinate system with x and y axes. A vertical dashed line represents a asymptote at $x = a$. A horizontal dashed line represents a asymptote at $y = \frac{1}{k}$. The function $y = \frac{1}{f(x)}$ is plotted as a blue curve that approaches the vertical asymptote as $x \rightarrow a$ and the horizontal asymptote as $x \rightarrow \infty$. The curve passes through the origin $(0,0)$.</p>

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(ii)	
(b)	<p><u>Method 1</u></p> <p>After 1st transformation:</p> $g(x) \rightarrow g\left(\frac{1}{2}x\right)$ <p>After 2nd transformation:</p> $g\left(\frac{1}{2}x\right) \rightarrow g\left(-\frac{1}{2}x\right)$ <p>After final transformation:</p> $g\left(-\frac{1}{2}x\right) \rightarrow 1 + g\left(-\frac{1}{2}x\right)$ $1 + g\left(-\frac{1}{2}x\right) = 1 - \frac{1}{x}$ $g\left(-\frac{x}{2}\right) = -\frac{1}{x} \Rightarrow g\left(\frac{x}{2}\right) = \frac{1}{2} \left(\frac{1}{\frac{x}{2}}\right)$ $g(x) = \frac{1}{2x}$

	<p><u>Method 2</u></p> <p>Let $h(x)$ be the expression after the final transformation.</p> <p>(a) Before final transformation:</p> $1 - \frac{1}{x} - 1 = -\frac{1}{x}$ <p>(b) Before 2nd transformation:</p> $-\left(-\frac{1}{x}\right) = \frac{1}{x}$ <p>Before 1st transformation (original expression)</p> $\frac{1}{2(x)} = \frac{1}{2x}$ $g(x) = \frac{1}{2x}$
7(i)	$u_2 = \frac{15}{16} \quad u_3 = \frac{63}{64} \quad u_4 = \frac{255}{256}$
(ii)	<p>Considering $1 - u_2 = \frac{1}{16}$ $1 - u_3 = \frac{1}{64}$ $1 - u_4 = \frac{1}{256}$</p> $\therefore 1 - u_n = \left(\frac{1}{2}\right)^{2n}$ <p>Hence, the conjecture is $u_n = 1 - \left(\frac{1}{2}\right)^{2n}$.</p> <p>Let P_n be the statement $u_n = 1 - \left(\frac{1}{2}\right)^{2n}$ for $n \in \mathbb{Z}^+$.</p> <p>When $n = 1$,</p> $\text{LHS} = u_1 = \frac{3}{4}$ $\text{RHS} = 1 - \left(\frac{1}{2}\right)^{2(1)} = \frac{3}{4} \quad \text{Since LHS} = \text{RHS}, P_1 \text{ is true.}$ <p>Assume that P_k is true for some $k \in \mathbb{Z}^+$.</p>

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i.e. assume $u_k = 1 - \left(\frac{1}{2}\right)^{2k}$, for some $k \in \mathbb{Z}^+$.

To prove that P_{k+1} is true,

i.e. prove $u_{k+1} = 1 - \left(\frac{1}{2}\right)^{2(k+1)}$.

$$\begin{aligned}u_{k+1} &= u_k + \frac{3}{4} \left(\frac{1}{2}\right)^{2k} \\&= 1 - \left(\frac{1}{2}\right)^{2k} + \frac{3}{4} \left(\frac{1}{2}\right)^{2k} \\&= 1 - \left(\frac{1}{2}\right)^{2k} \left(1 - \frac{3}{4}\right) \\&= 1 - \left(\frac{1}{2}\right)^{2k} \left(\frac{1}{4}\right) \\&= 1 - \left(\frac{1}{2}\right)^{2k+2}\end{aligned}$$

Hence, P_k is true $\Rightarrow P_{k+1}$ is true.

Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for **all** $n \in \mathbb{Z}^+$.

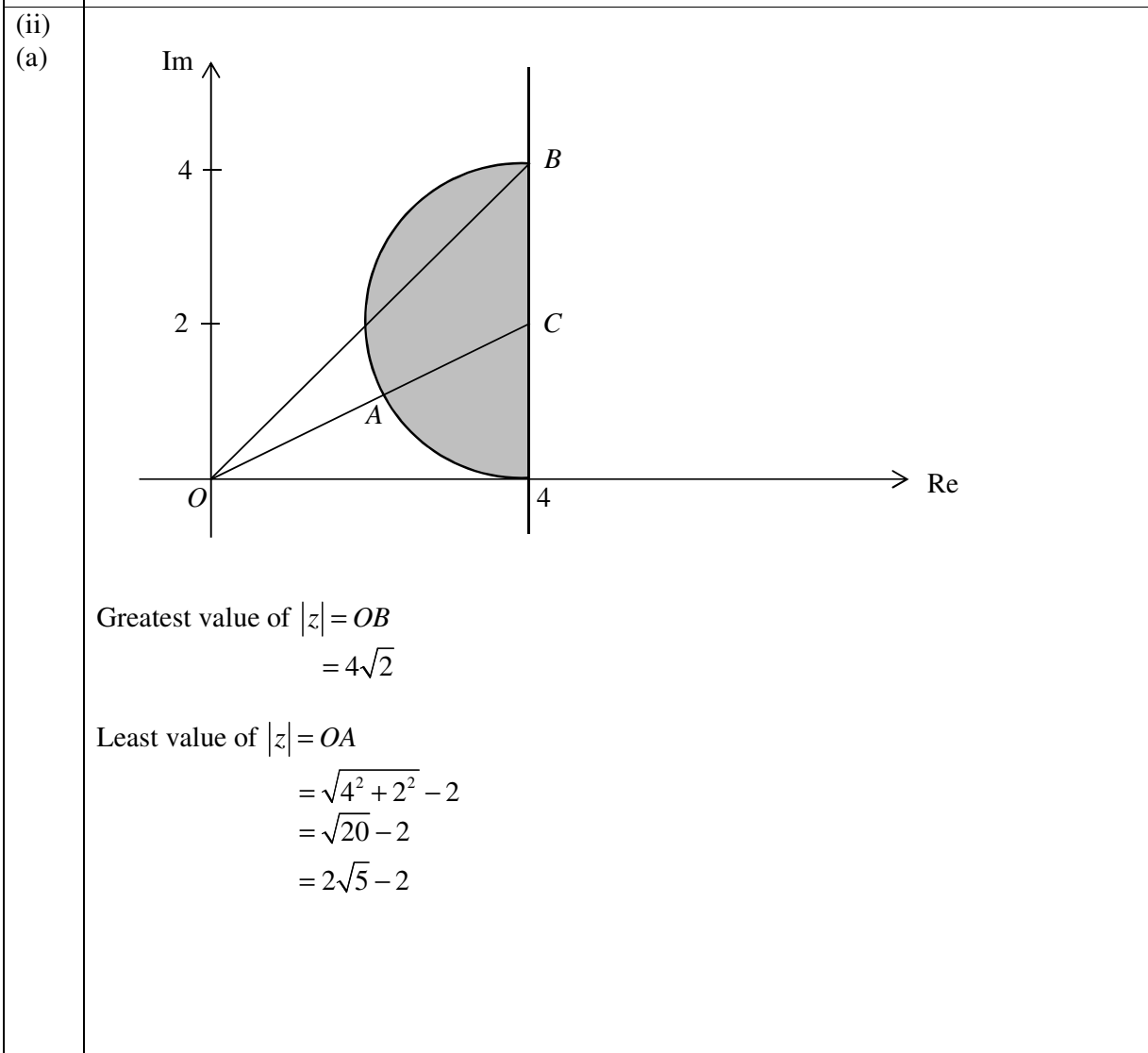
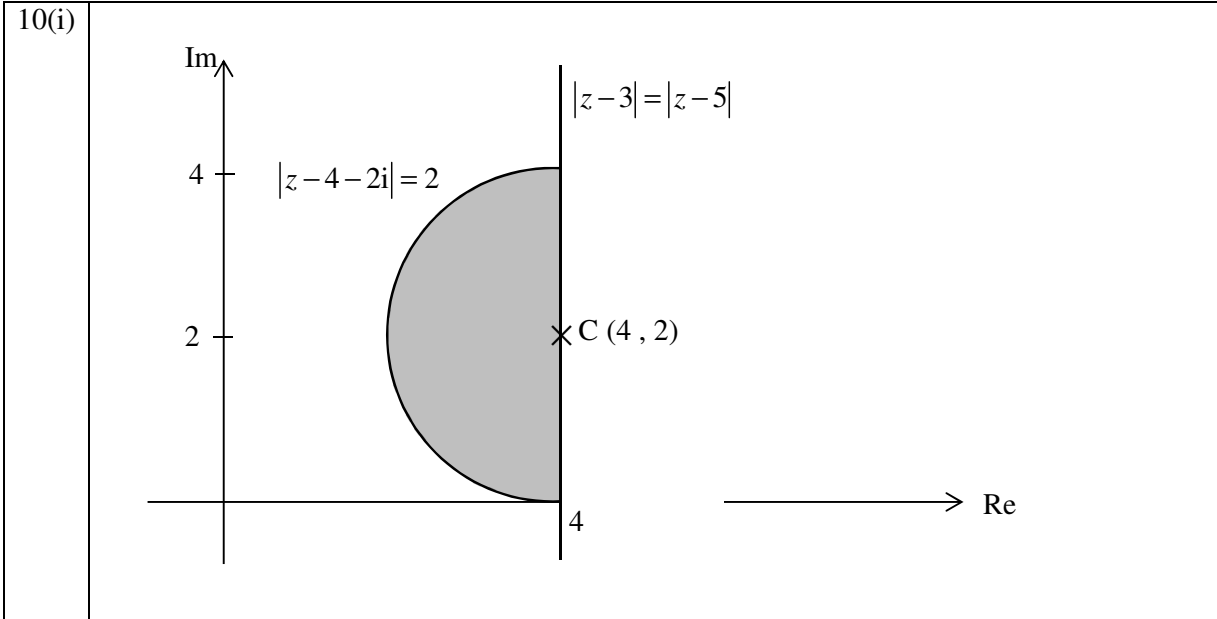
(iii)	$\sum_{r=2}^N \frac{3}{4} \left(\frac{1}{2}\right)^{2r} = \sum_{r=2}^N (u_{n+1} - u_n)$ $= u_3 - u_2$ $+ u_4 - u_3$ $+ u_5 - u_4$ \vdots $+ u_N - u_{N-1}$ $+ u_{N+1} - u_N$ $= u_{N+1} - u_2$ $= \left(1 - \left(\frac{1}{2}\right)^{2(N+1)}\right) - \left(1 - \left(\frac{1}{2}\right)^{2(2)}\right)$ $= \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^{2N+2}$ $= \left(\frac{1}{4}\right) \left(\frac{1}{4} - \left(\frac{1}{2}\right)^{2N}\right)$								
(iv)	$\left(\frac{1}{4}\right) \left(\frac{1}{4} - \left(\frac{1}{2}\right)^{2N}\right) > \frac{3}{50}$ <p>Using GC Table,</p> <table border="1" data-bbox="272 1115 1084 1325"> <thead> <tr> <th>N</th> <th>$\left(\frac{1}{4}\right) \left(\frac{1}{4} - \left(\frac{1}{2}\right)^{2N}\right)$</th> </tr> </thead> <tbody> <tr> <td>3</td> <td>0.05859</td> </tr> <tr> <td>4</td> <td>0.06152</td> </tr> <tr> <td>5</td> <td>0.06226</td> </tr> </tbody> </table> <p>Therefore, the smallest integer value of N is 4.</p>	N	$\left(\frac{1}{4}\right) \left(\frac{1}{4} - \left(\frac{1}{2}\right)^{2N}\right)$	3	0.05859	4	0.06152	5	0.06226
N	$\left(\frac{1}{4}\right) \left(\frac{1}{4} - \left(\frac{1}{2}\right)^{2N}\right)$								
3	0.05859								
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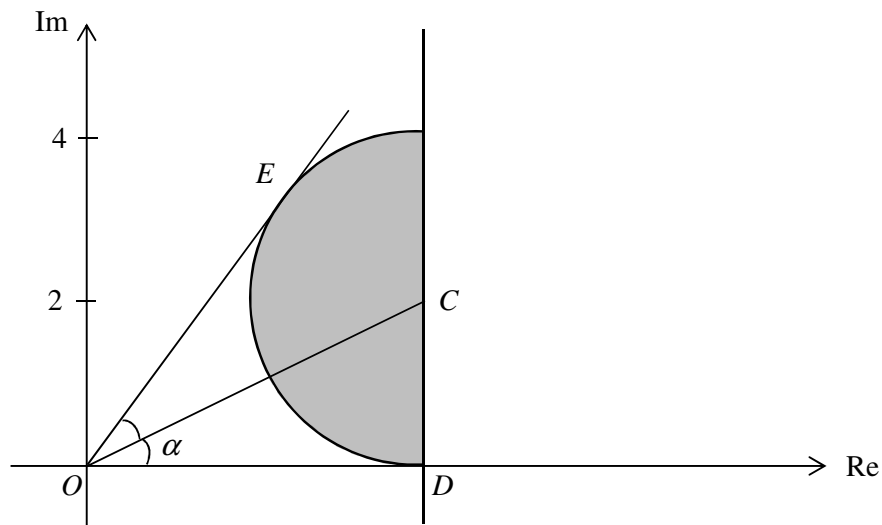
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<p>8(a)</p>	$\int_5^{p+9} \frac{1}{\sqrt{9-x}} dx = \int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}} dx$ $\int_5^{p+9} (9-x)^{-\frac{1}{2}} dx = \frac{1}{2} \int_0^{\frac{1}{4}} \frac{2}{\sqrt{1-(2x)^2}} dx$ $\left[-2(9-x)^{\frac{1}{2}} \right]_5^{p+9} = \frac{1}{2} \left[\sin^{-1} 2x \right]_0^{\frac{1}{4}}$ $\left[-2(-p)^{\frac{1}{2}} + 2(9-5)^{\frac{1}{2}} \right] = \frac{1}{2} \sin^{-1} \frac{1}{2}$ $4 - 2(-p)^{\frac{1}{2}} = \frac{1}{2} \left(\frac{\pi}{6} \right)$ $2(-p)^{\frac{1}{2}} = 4 - \left(\frac{\pi}{12} \right)$ $p = - \left[2 - \left(\frac{\pi}{24} \right) \right]^2$ $= - \left(\frac{48 - \pi}{24} \right)^2$
<p>(b)</p>	$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}} dx$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos \theta}{\sqrt{1-\cos^2 \theta}} (-2 \cos \theta \sin \theta) d\theta$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos \theta}{\sqrt{\sin^2 \theta}} (-2 \cos \theta \sin \theta) d\theta$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} -2 \cos^2 \theta d\theta$ $= - \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} (\cos 2\theta + 1) d\theta$ $= - \left[\frac{1}{2} \sin 2\theta + \theta \right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}$ $= - \frac{1}{2} \left[\left(\sin \frac{\pi}{2} + \frac{\pi}{4} \right) - \left(\sin \pi + \frac{\pi}{2} \right) \right]$ $= - \frac{1}{2} \left(1 + \frac{\pi}{4} - \frac{\pi}{2} \right)$ $= - \frac{1}{2} + \frac{\pi}{4}$

<p>9(i)</p>	$\overrightarrow{OP} = \frac{2}{5} \mathbf{a}$ <p>Since $OPQB$ is a parallelogram,</p> $\overrightarrow{OB} = \overrightarrow{PQ}$ $\mathbf{b} = \overrightarrow{OQ} - \overrightarrow{OP}$ $\mathbf{b} = \overrightarrow{OQ} - \frac{2}{5} \mathbf{a}$ $\overrightarrow{OQ} = \frac{2}{5} \mathbf{a} + \mathbf{b}$
<p>(ii)</p>	<p>Area of triangle OAQ</p> $= \frac{1}{2} \overrightarrow{OA} \times \overrightarrow{OQ} $ $= \frac{1}{2} \left \mathbf{a} \times \left(\frac{2}{5} \mathbf{a} + \mathbf{b} \right) \right $ $= \frac{1}{2} \left \mathbf{a} \times \frac{2}{5} \mathbf{a} + \mathbf{a} \times \mathbf{b} \right $ $= \frac{1}{2} \mathbf{a} \times \mathbf{b} $ <p>Therefore, k is $\frac{1}{2}$.</p>
<p>(iii)</p>	<p>$OPB : OAB$</p> <p>2 : 5</p>
<p>(iv)</p>	<p>Since $\mathbf{a} \times \mathbf{b}$ is a unit vector,</p> $ \mathbf{a} \times \mathbf{b} = \mathbf{a} \mathbf{b} \sin \theta$ $1 = \mathbf{a} \mathbf{b} \sin \theta$ $1 = 2 \mathbf{b} \sin 60^\circ$ <p>Therefore, $\mathbf{b} = \frac{1}{\sqrt{3}}$</p>

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Least value of $\arg(z) = 0$

$$\tan \alpha = \frac{2}{4}$$

$$\alpha = 0.46364$$

Note that $\angle COE = \alpha$

Hence, greatest value of $\arg(z) = 2\alpha$
 $= 0.927$

11
(i)

$$x \cos 2x = 0$$

$$x = 0 \quad \text{or} \quad \cos 2x = 0$$

$$2x = \cos^{-1} 0$$

$$= \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} \quad \text{for} \quad 0 \leq 2x \leq 2\pi$$

$$x = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4} \quad \text{for} \quad 0 \leq x \leq \pi$$

Therefore the x -intercepts are $x = 0$, $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$

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(ii)	<p style="text-align: center;">$y = x \cos 2x$</p> <p style="text-align: right;">(π, π)</p>
(iii)	$\int x \cos 2x \, dx$ $= \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, dx$ $= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c$ $\int_{\frac{\pi}{4}}^{\pi} x \cos 2x \, dx$ $= -\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x \cos 2x \, dx + \int_{\frac{3\pi}{4}}^{\pi} x \cos 2x \, dx$ $= -\left[\frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \left[\frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]_{\frac{3\pi}{4}}^{\pi}$ $= \frac{7\pi}{8} + \frac{1}{4}$
(iv)	$\int_{\frac{3\pi}{4}}^{\pi} \pi (x \cos 2x)^2 \, dx$ ≈ 10.465 $= 10.5 \text{ (3 s.f.)}$

12

(a)

$$\frac{1}{3}\pi r^2 h = 50\pi \text{ cm}^3$$

$$r^2 h = 150$$

$$h = \frac{150}{r^2}$$

$$h^2 + r^2 = l^2$$

$$A = \pi r l$$

$$A^2 = \pi^2 r^2 l^2$$

$$= \pi^2 r^2 (h^2 + r^2)$$

$$= \pi^2 r^2 h^2 + \pi^2 r^4$$

$$= \pi^2 r^2 \left(\frac{150}{r^2}\right)^2 + \pi^2 r^4$$

$$= \frac{22500\pi^2}{r^2} + \pi^2 r^4$$

Differentiating with respect to x :

$$2A \frac{dA}{dr} = -\frac{2(22500)\pi^2}{r^3} + 4\pi^2 r^3$$

Since $\frac{dA}{dr} = 0$,




$$0 = -\frac{2(22500)\pi^2}{r^3} + 4\pi^2 r^3$$

$$\frac{2(22500)\pi^2}{r^3} = 4\pi^2 r^3$$

$$r^6 = 11250$$

$$r = 4.7336 \approx 4.73$$

$$h = 6.694 \approx 6.69$$

r	4.73^-	4.73	4.73^+
$\frac{dA}{dr}$	-ve	0	+ve
Slope			

Therefore, $r = 4.73$ (3s.f) and $h = 6.69$ (3 s.f) require the least amount of material.

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(bi) Given $V = \frac{\pi h^3}{12}$,

$$\frac{dV}{dh} = \frac{3\pi h^2}{12} = \frac{\pi h^2}{4}$$

When $h = 3$, $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{4}{\pi(2)^2} \times -3 = -\frac{3}{\pi}$ (or -0.955)

The rate at which the depth is decreasing at the instant when the depth is 2 cm is $\frac{3}{\pi}$ cms⁻¹.

Alternative Method

Given $V = \frac{\pi h^3}{12}$,

$$\begin{aligned}\frac{dV}{dt} &= \frac{3\pi h^2}{12} \frac{dh}{dt} \\ &= \frac{\pi(2)^2}{4} (-3) \\ &= -\frac{3}{\pi}\end{aligned}$$

The rate at which the depth is decreasing at the instant when the depth is 2 cm is $\frac{3}{\pi}$ cms⁻¹.

(ii) Change in volume = $\frac{\pi(6^3 - 3^3)}{12} = \frac{189\pi}{12}$ cm³

Time taken = $\frac{189\pi}{12} \div 3 = \frac{189\pi}{36}$ s (or 16.5s (3s.f))