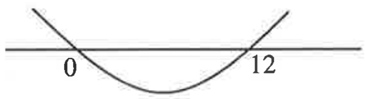
Solutions to Nov 2013 Mathematics H1 (8864)

1. $x^2-(k-2)x+(2k+1)=0$ has no real roots

$$[-(k-2)]^2-4(1)(2k+1)<0$$

$$k^2-12k<0$$

$$k(k-12)<0$$



$$0<k<12$$

Set of values of k is $\{k\in\mathbb{R}:0<k<12\}$

2.

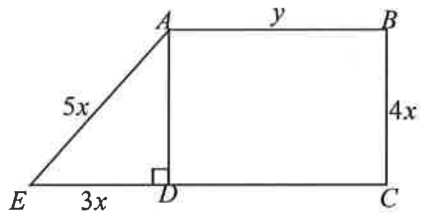
(i) $\dfrac{d}{dx}\ln(1+2x^2)=\dfrac{4x}{1+2x^2}$

(ii) $\displaystyle\int_{-1}^{0}\frac{1}{(1-3x)^4}\,dx=\left[\frac{(1-3x)^{-3}}{(-3)(-3)}\right]_{-1}^{0}$

$$=\frac{1}{9}\left[\frac{1}{(1-3x)^3}\right]_{-1}^{0}=\frac{1}{9}\left(1-\frac{1}{4^3}\right)$$

$$=\frac{1}{9}\times\frac{63}{64}=\frac{7}{64}$$

3.



(i) $AE=5x$ (pythagoras theorem)

$5x+2y+3x+4x=20$

$\therefore y=10-6x$ ....(1)

$$S=\frac{1}{2}(3x)(4x)+4xy$$

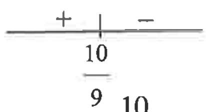
$$=6x^2+4x(10-6x)$$

$$=-18x^2+40x$$

(ii) $\dfrac{dS}{dx}=-36x+40$

$\dfrac{dS}{dx}=0\Rightarrow x=\dfrac{40}{36}=\dfrac{10}{9}$

Sign of $\dfrac{dS}{dx}$



$\therefore S$ is maximum when $x=\dfrac{10}{9}$

Maximum value of

$$S=-18\left(\frac{10}{9}\right)^2+40\left(\frac{10}{9}\right)=\frac{200}{9}$$

4. $C: y=x^3-ax^2+3x+6$

(i) $\dfrac{dy}{dx}=3x^2-2ax+3$

At P where $x=1$, $\dfrac{dy}{dx}=3-2a+3=6-2a$

$\therefore$ gradient of normal at $P=-\dfrac{1}{6-2a}$

(ii) When $x=1$, $y=1-a+3+6=10-a$

Equation of normal at P is

$$y-(10-a)=\frac{1}{2a-6}(x-1)$$

The normal passes through $(-5,3)$,

$$\therefore 3-10+a=\frac{-6}{2a-6}$$

$$(a-7)(2a-6)+6=0$$

$$2a^2-20a+48=0\Rightarrow a^2-10a+24=0$$

Hence a satisfies the equation $a^2-10a+24=0$

$$a^2-10a+24=0$$

$$\Rightarrow(a-4)(a-6)=0$$

$$\therefore a=4 \text{ or } 6$$

(iii) $a=4$

Equation of normal at P is $y-6=\dfrac{1}{2}(x-1)$

i.e. $2y-x=11$ .... (1)

$y=x$ ....(2)

Solving (1) & (2): $x=11,\ y=11$

Coordinates of point of intersection is $(11,11)$.

5(i) $e^{2-2x}=2e^{-x}$

$\therefore 2-2x=\ln 2-x$

$\Rightarrow x=2-\ln 2$

(ii) $y=e^{2-2x}-2e^{-x}$

$$\frac{dy}{dx}=-2e^{2-2x}+2e^{-x}=-2\left(e^{2-2x}-e^{-x}\right)$$

$$\frac{dy}{dx}=0\Rightarrow e^{2-2x}-e^{-x}=0$$

$$\Rightarrow e^2e^{-2x}-e^{-x}=0$$

$$\Rightarrow e^{-x}\left(e^2e^{-x}-1\right)=0$$

$\therefore e^2e^{-x}-1=0$ since $e^{-x}\neq 0$

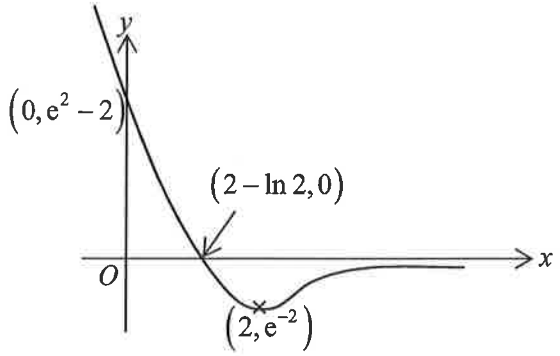
$$\Rightarrow e^{-x}=e^{-2}\Rightarrow x=2$$

$$\therefore y=e^{2-4}-2e^{-2}=-e^{-2}$$

$\therefore C$ has a stationary point at $(2,-e^{-2})$

5(iii) When $y=0,\ e^{2-2x}-2e^{-x}=0 \Rightarrow x=2-\ln 2$

When $x=0,\ y=e^2-2$



(iv) Since $1<2-\ln 2$,

$\therefore$ required area $= \int_0^1 e^{2-2x}-2e^{-x}\,dx$

$\approx 1.93 \text{ unit}^2$

6.

Ticket Value	Number of tickets	Number to be selected
\$X	5000	$\frac{5000}{30000}\times 150 = 25$
\$Y	10000	$\frac{10000}{30000}\times 150 = 50$
\$Z	15000	$\frac{15000}{30000}\times 150 = 75$
	30000	150

(i) Suky will select the sample of 150 from each of the 3 groups comprising 25 of the \$*X*, 50 of the \$*Y* and 75 of the \$*Z* ticket holders. Simple random sampling will be used to select each of the 3 subgroups. To select the 25 tickets from the 5000 \$*X* tickets, the 5000 tickets are each assigned a number from 1 to 5000. Then 25 numbers are generated using a random number generator and the corresponding 25 ticket holders will be selected. The procedure is repeated for the selection of the 50 and 75 ticket holders.

(ii) The 3 subgroups are represented proportionally in the sample as they are in the population that is in the ratio of 1:2:3.

7.

Let $w=t-75$, then $\sum w=305$ and $\sum w^2=29555$

Unbiased estimate of population mean is

$$\bar{t}=\bar{w}+75=\frac{305}{250}+75=76.22$$

Unbiased estimate of population variance is

$$s_t^2=s_w^2=\frac{1}{249}\left[29555-\frac{305^2}{250}\right]\approx 117.20\approx 117$$

Let μ hours be the population mean of *T*.

$H_0: \mu=75$

$H_1: \mu>75$

Level of significance: $2\frac{1}{2}\%$

Test Statistic: When H_0 is true,

$$Z=\frac{\bar{T}-75}{s_t/\sqrt{250}} \sim N(0,1) \text{ approximately}$$

Computation: $n=250,\ \bar{t}=76.22,\ s_t^2=117.20$

$z\approx 1.7818,\ p-\text{value}\approx 0.037389$

Conclusion: Since *p*-value = 0.0374> 0.025, H_0 is not rejected at 2.5% significance level. There is insufficient evidence to conclude that the population mean is more than 75 hours. Hence the claim is not justified at the 2.5% significance level.

8. Let *X* be the number of batteries, out of 10, with a lifetime of at least 100 hours.

Then $X \sim B(10, 0.8)$

(i) Required probability = $P(X=10)$

$\approx 0.10737 \approx 0.107$

(ii) Required probability = $P(X\geq 8)$

$=1-P(X\leq 7)$

$\approx 0.67780 \approx 0.678$

(iii) Let *Y* be the number of packs in a batch of 80 packs that will satisfy the customer.

$Y \sim B(80, 0.67780)$

Since *n*=80 is sufficiently large such that $np=54.224>5$ and $n(1-p)=25.776>5$, use normal approximation.

$E(Y)=np=54.224$

$Var(Y)=np(1-p)=17.47097$

$Y \sim N(54.224, 17.47097)$ approximately.

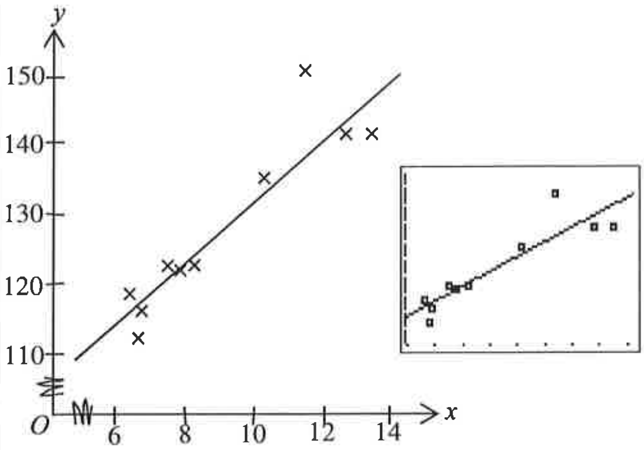
$P\left(Y\geq \frac{75}{100}\times 80\right)$

$=P(Y\geq 60)$

$=P(Y>59.5)$ (using continuity correction)

$\approx 0.10343 \approx 0.103$

9(i)



(ii) The product moment correlation coefficient, $r \approx 0.90326 \approx 0.903$

As $r \approx 0.903$ is close to $+1$ and points in scatter diagram seem to lie close to a straight line with positive gradient, these are indications of a strong positive linear relationship between the ages (x) and the heights (y). This means that as x increases, y tends to increase at a constant rate.

(iii) Equation of the regression line of y on x is

$y \approx 87.4311 + 4.4617x \approx 87.43 + 4.46x$

(iv) When $x = 13.2$,

$y \approx 87.4311 + 4.4617 \times 13.2 \approx 146.33$

Estimated height is 146 cm.

The estimate is reliable since the linear model is appropriate as seen in (ii) and the value 13.2 lies within the data range of $6.6 \le x \le 13.5$. The line is suitable for interpolation.

10.

Let X g be the mass of salt in a bottle of the sauce and μ g be the population mean of X.

Then $X \sim \mathrm{N}(\mu, 0.8^2)$

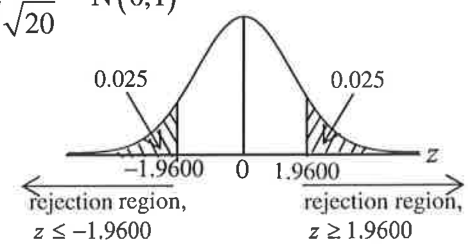
$\mathrm{H_0}: \mu = 12$

$\mathrm{H_1}: \mu \neq 12$

Level of significance:5%

Test Statistic: When $\mathrm{H_0}$ is true,

$$Z = \frac{\overline{X} - 12}{0.8/\sqrt{20}} \sim \mathrm{N}(0,1)$$



Rejection region: $z \le -1.9600$ or $z \ge 1.9600$

Computation: $n = 20,\ \overline{x} = m,\ \sigma = 0.8$

$$\therefore z = \frac{m - 12}{0.8/\sqrt{20}}$$

Company's claim is accepted

$\Rightarrow \mathrm{H_0}$ is not rejected

$\therefore -1.9600 < z < 1.9600$

$$\Rightarrow -1.9600 < \frac{m-12}{0.8/\sqrt{20}} < 1.9600$$

$$\therefore 12 - 1.9600 \times \frac{0.8}{\sqrt{20}} < m < 12 + 1.9600 \times \frac{0.8}{\sqrt{20}}$$

$\Rightarrow 11.649 < m < 12.351$

Required set is $\{m \in \mathbb{R} : 11.6 < m < 12.4\}$

Alternatively,

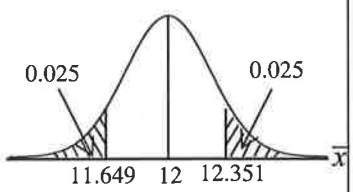
Test Statistic: When $\mathrm{H_0}$ is true, $\overline{X} \sim \mathrm{N}\left(12, \frac{0.8^2}{20}\right)$

From GC:

$\mathrm{P}(\overline{X} \le 11.649) = 0.025$

$\mathrm{P}(\overline{X} \le 12.351) = 0.975$

$\therefore 11.649 < m < 12.351$



Required set is $\{m \in \mathbb{R} : 11.6 < m < 12.4\}$

Let Y g be the mass of salt in a bottle of the new variety of sauce and μ_1 g be the population mean.

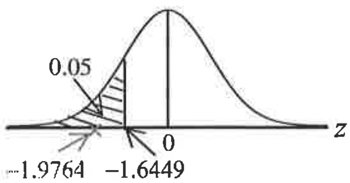
$\mathrm{H_0}: \mu_1 = 12$

$\mathrm{H_1}: \mu_1 < 12$

Level of significance:5%

Test Statistic: When $\mathrm{H_0}$ is true,

$$Z = \frac{\overline{Y} - 12}{0.8/\sqrt{40}} \sim \mathrm{N}(0,1)$$



Computation: $n = 40,\ \overline{y} = 11.75,\ \sigma = 0.8$

$z \approx -1.9764,\ p - \text{value} \approx 0.024053$

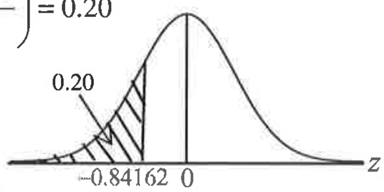
Conclusion: Since $p - \text{value} \approx 0.0241 < 0.05$, $\mathrm{H_0}$ is rejected at 5% significance level. So there is sufficient evidence to conclude that the population mean mass of salt of the new variety of sauce is less than 12g. Hence the company's claim is justified.

11. Let X g be the mass of the content in a type A packet of animal food.

$X \sim \mathrm{N}(1000, \sigma^2)$

(i) $\mathrm{P}(X < 990) = 0.20$

$\mathrm{P}\left(Z < \frac{990-1000}{\sigma}\right) = 0.20$



From GC:

$\mathrm{P}(Z < -0.84162) = 0.20$

$\therefore \frac{990-1000}{\sigma} = -0.84162$

$\Rightarrow \sigma = \frac{10}{0.84162} \approx 11.882 \approx 11.9$

(ii) Let U g and V g be the mass of one scoop of ingredient P and Q respectively.

Then $U \sim \mathrm{N}(240, 10^2)$ and $V \sim \mathrm{N}(145, 8^2)$

Let Y g be the mass of the content in a type B packet of animal food.

Then $Y = U_1 + U_2 + U_3 + V_1 + V_2$

$\mathrm{E}(Y) = 3\mathrm{E}(U) + 2\mathrm{E}(V)$

$= 3 \times 240 + 2 \times 145 = 1010$

$\mathrm{Var}(Y) = 3\mathrm{Var}(U) + 2\mathrm{Var}(V)$

$= 3 \times 10^2 + 2 \times 8^2 = 428$

$Y \sim \mathrm{N}(1010, 428)$

$\mathrm{P}(Y < 1000) \approx 0.31442 \approx 0.314$

(iii) $\mathrm{E}(Y - X) = \mathrm{E}(Y) - \mathrm{E}(X) = 1010 - 1000 = 10$

$\mathrm{Var}(Y - X) = \mathrm{Var}(Y) + \mathrm{Var}(X)$

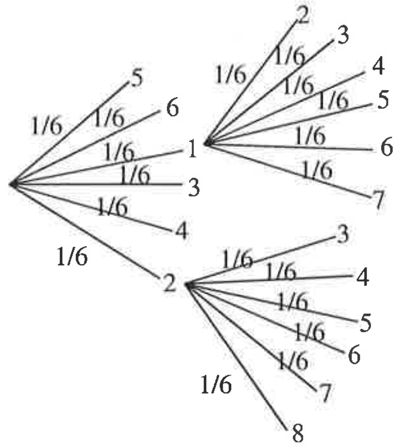
$= 428 + 11.882^2 \approx 569.18$

$Y - X \sim \mathrm{N}(10, 569.18)$

$\mathrm{P}(Y > X) = \mathrm{P}(Y - X > 0)$

$\approx 0.66245 \approx 0.662$

12(i)



(ii) $\mathrm{P}(A) = 2 \times \frac{1}{6} + 4\left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$

(iii) $\mathrm{P}(A \cap B) = 2\left(\frac{1}{6} \times \frac{1}{6}\right) + 2\left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{9}$

(iv) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= \frac{4}{9} + \frac{2}{6} - \frac{1}{9} = \frac{2}{3}$

(v) $\mathrm{P}(B \mid A') = \frac{\mathrm{P}(B \cap A')}{\mathrm{P}(A')}$

$= \frac{\mathrm{P}(B) - \mathrm{P}(A \cap B)}{1 - \mathrm{P}(A)}$

$= \frac{\frac{2}{6} - \frac{1}{9}}{1 - \frac{4}{9}} = \frac{2}{9} \times \frac{9}{5}$

$= \frac{2}{5}$