2012 NYJC H2 Math Preliminary exam Paper 2 solutions

1 (a) $zw = (1+\mathrm{i}a)(-b-\mathrm{i}) = -b - \mathrm{i} - ab\mathrm{i} + a = 6 - 11\mathrm{i}$

$a - b = 6$

$1 + ab = 11$

$\Rightarrow 1 + a(a-6) = 11$

$a^2 - 6a - 10 = 0$

$a = \dfrac{6 \pm \sqrt{36+40}}{2} = \dfrac{6 \pm \sqrt{76}}{2} = 3 \pm \sqrt{19}$

Since $a > 0$, $a = 3 + \sqrt{19}$, $b = -3 + \sqrt{19}$

(b) $|u| = 2, \arg u = -\dfrac{2}{3}\pi$ $\quad |v| = 5, \arg v = \dfrac{3}{4}\pi$

$\left|\dfrac{v}{u^2}\right| = \dfrac{5}{4}$

$\arg\left(\dfrac{v}{u^2}\right) = \arg v - 2\arg u = \dfrac{3}{4}\pi + \dfrac{4}{3}\pi = \dfrac{25}{12}\pi$

$\therefore \arg\left(\dfrac{v}{u^2}\right) = \dfrac{1}{12}\pi$

$\left(\dfrac{v}{u^2}\right)^n = \left(\dfrac{5}{4}\right)^n\left(\cos\dfrac{n\pi}{12} + \mathrm{i}\sin\dfrac{n\pi}{12}\right)$

Since $\left(\dfrac{v}{u^2}\right)^n$ is imaginary, $\cos\dfrac{n\pi}{12} = 0$

$\Rightarrow \dfrac{n\pi}{12} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$

Since Im part of $\left(\dfrac{v}{u^2}\right)^n$ is negative, $\sin\dfrac{n\pi}{12} < 0$

$\Rightarrow \dfrac{n\pi}{12} = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, \ldots$

Hence, smallest $n = 18$

Alternatively,

Since $\left(\dfrac{v}{u^2}\right)^n$ is imaginary and negative,

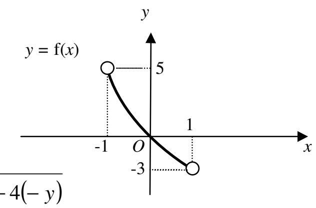
$\dfrac{n\pi}{12} = -\dfrac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$

$n = -6 + 24k$

To get smallest positive n, let $k = 1$.

Hence, smallest $n = 18$.

2 **(i)** Clearly f is 1-1 on the domain $(-1,1)$ since every horizontal line $y = k$ ($k \in \mathbb{R}$) cuts the graph at most once. So f^{-1} exists.



Let $y = x^2 - 4x$.

Rearranging, we have $x^2 - 4x - y = 0$.

Solving for x, we obtain $x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(-y)}}{2}$

$= 2 \pm \sqrt{4+y}$.

Since $|x| < 1$, $x = 2 - \sqrt{4+y}$ as $2 + \sqrt{4+y} \geq 2$.

So $\mathrm{f}^{-1}(y) = x = 2 - \sqrt{4+y}$.

Hence, $\mathrm{f}^{-1} : x \to 2 - \sqrt{4+x}$, $x \in \mathbb{R}$, $-3 < x < 5$.

(ii) Since $\mathrm{R_f} = (-3,5) \subset \mathbb{R}\backslash\{-3\} = \mathrm{D_g}$, gf exists.

(iii) $\mathrm{gf}(x) = \mathrm{g}(x^2 - 4x) = 1 + \dfrac{1}{(x^2-4x)+3}$

$= 1 + \dfrac{1}{x^2 - 4x + 3}$

(iv) $\mathrm{h}'(x) = -\dfrac{2x-4}{(x^2-4x+3)^2} = \dfrac{4-2x}{(x^2-4x+3)^2}$

Given $-1 < x < 1 \Rightarrow -2 < -2x < 2 \Rightarrow 2 < 4 - 2x < 6$.

Since $4 - 2x > 0$ *and* $(x^2-4x+3)^2 > 0$, $\mathrm{h}'(x) > 0$ and so h is increasing on $\mathrm{D_h}$.

Note that $\mathrm{h}(-1) = 1 + \dfrac{1}{1+4+3} = \dfrac{9}{8}$ and

$\mathrm{h}(x) \to \infty$ as $x \to 1^-$.

Hence $\mathrm{R_h} = \left(\dfrac{9}{8}, \infty\right)$ or $(1.125, \infty)$.

3

$x^2 + y^2 = \frac{1}{4}$.

Let V be the volume of the silo.

$V = \pi x^2 y$

$= \pi(\frac{1}{4} - y^2)y$

$= \pi(\frac{1}{4}y - y^3)$ km^3

$\frac{dV}{dy} = \pi(\frac{1}{4} - 3y^2) = 0$

$\Rightarrow y = \frac{1}{2\sqrt{3}}$ (reject $-\frac{1}{2\sqrt{3}}$)

$\frac{d^2V}{dy^2} = -6\pi y < 0$ when $y = \frac{1}{2\sqrt{3}}$.

maximum $V = \pi(\frac{1}{4} - \frac{1}{12})\frac{1}{2\sqrt{3}} = \pi\frac{1}{12\sqrt{3}}$ or $\pi\frac{\sqrt{3}}{36}$ km^3

Let V_d be the volume of the dome. $V_d = \frac{2}{3}\pi r^3 \Rightarrow \frac{dV_d}{dt} = 2\pi r^2 \frac{dr}{dt}$

$\Rightarrow 0.75 = = 2\pi r^2 (\frac{3}{4\pi})$

$\Rightarrow r^2 = 0.5 \Rightarrow r = 1/\sqrt{2}$ km

$A = 2\pi r^2$

$\frac{dA}{dt} = 4\pi r \frac{dr}{dt}$

$= 4\pi(\frac{1}{\sqrt{2}})\frac{3}{4\pi}$

$= \frac{3\sqrt{2}}{2}$ km^{2}s^{-1}

4

(a)

Area R

$= 2\ln 2 - \int_{-1}^{1} \ln(2-|x|)\, dx$ or $2\ln 2 - 2\int_0^1 \ln(2-|x|)\, dx$

$= 2\ln 2 - 2\int_0^1 \ln(2-x)\, dx$ since $x > 0$

$= 2\ln 2 - 2\left\{\left[x\ln(2-x)\right]_0^1 + \int_0^1 \frac{x}{2-x} dx\right\}$

$= 2\ln 2 - 2\left\{0 + \int_0^1 -1 + \frac{2}{2-x} dx\right\}$

$= 2\ln 2 - 2\left[-x - 2\ln(2-x)\right]_0^1$

$= 2\ln 2 - 2(-1 + 2\ln 2)$

$= 2 - 2\ln 2$

(b) Volume needed

$= \pi\int_0^1 y^2\, dx$

When $x = 0$, $0 = (1+t)^{\frac{2}{3}} \Rightarrow t = -1$

When $x = 1$, $1 = (1+t)^{\frac{2}{3}} \Rightarrow 1 + t = \pm 1$

$t = -2$ or 0 (rejected out of range)

$x = (1+t)^{\frac{2}{3}}$

$\frac{dx}{dt} = \frac{2}{3}(1+t)^{-\frac{1}{3}}$

Thus, volume needed

$= \pi\int_{-1}^{-2} \left[\ln(t^2)\right]^2 \frac{2}{3}(1+t)^{-\frac{1}{3}}\, dt$

= 1.80 units^3

5

Either stratified sampling or quota sampling method can be accepted as appropriate methods. Description -

Stratified–

i) states that sampling frame is obtained listing the participants according to the types of educational institutions that they come from

ii)states that using simple random sampling, 35, 25, 15 and 5 participants are selected from each of the strata (primary, secondary, JC, poly) respectively

Advantage – participants from all the 4 institutions are represented proportionately

Responses can be analysed by the institutions that participants come from

Disadvantage – relatively inconvenient to carry out

Quota –

States/sets the quota to be selected from each institution type, such that they add up to 80 in total (e.g. 30 primary, 20 secondary, 20 JC, 10 poly)

Selects from a list the participants for the survey, in accordance with the quota set above.

Advantage – easy to choose the 80 participants

Disadvantage – non-random method, so results obtained may be biased (e.g. the list may cluster participants from the same school)

6

(i) ${}^{6}C_2 \times {}^{5}C_4 = 75$ ways

(ii) Number of ways if at least one of the sisters are included
= number of ways without restriction – number of ways if none of the sisters is included
$= {}^{11}C_6 - {}^{8}C_6 = 434$

Or ${}^{3}C_1 \times {}^{8}C_5 + {}^{3}C_2 \times {}^{8}C_4 + {}^{3}C_3 \times {}^{8}C_3 = 434$

(iii) Select a man to be between the 2 sisters and group the 3 of them as one unit and arrange 4 units round a table

$${}^{3}C_1 \times 3! \times 2 = 36$$

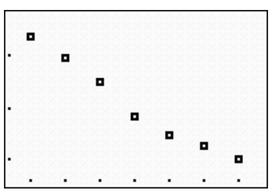
(iv)First arrange the other 4 persons round the table. There are 4 ways to insert the sisters.

$$3! \times 4 = 24$$

or ${}^{4}C_2 \times 2! \times 2! = 24$

7

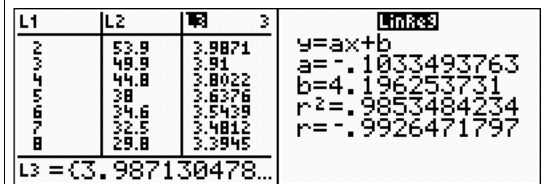
(i)By G.C., we obtain a scatter plot of the diagram as follows:



(ii)Although from the scatter diagram, it seems that there is a negative linear relationship between y and x within the given range. However, we note that the trade-in value of the car cannot continue to decrease linearly to become a negative value eventually. Thus, a linear model may not be appropriate.

(iii)Under a quadratic model, the trade-in value of the car would eventually increase over time, which would not make sense.

(iv)By G.C., we transform the y values as follows:



Thus, we have $\ln y = -0.103x + 4.196$.

When $x = 5.5$, $y = 37.631$. Thus, the trade-in value is $37631.

8

(i) P(first red bead is obtained on or before the 5th draw)

$$= \frac{2}{5} + \frac{3}{5}.\frac{2}{5} + \left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right) \text{ or } \sum_{r=0}^{4}\left(\frac{3}{5}\right)^r\left(\frac{2}{5}\right)$$

$= 0.922$

Or 1 – P(no red on first 5 draws) = $1 - \left(\frac{3}{5}\right)^5 = 0.922$

(ii) P(obtaining a first green bead on the 8th draw given that no green bead has been obtained after 5 draws) = P(red on 6th and 7th draws and green on 8th draw)=

$\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right) = 0.096$

(iii) P(exactly r draws are required for beads of both colours to be obtained)

$$= \left(\frac{2}{5}\right)^{r-1}\left(\frac{3}{5}\right) + \left(\frac{3}{5}\right)^{r-1}\left(\frac{2}{5}\right)$$

	$= \left(\frac{2}{5}\right)^{r-2}\left(\frac{6}{25}\right)+\left(\frac{3}{5}\right)^{r-2}\left(\frac{6}{25}\right)$ $= \left(\frac{6}{25}\right)\left[\left(\frac{2}{5}\right)^{r-2}+\left(\frac{3}{5}\right)^{r-2}\right]$, where $r = 2,3,4,\ldots$ (iv) P(first obtaining beads of different colours after 5 or more draws) $=\left(\frac{6}{25}\right)\left[\left(\left(\frac{2}{5}\right)^3+\left(\frac{3}{5}\right)^3\right)+\left(\left(\frac{2}{5}\right)^4+\left(\frac{3}{5}\right)^4\right)+\ldots\right]$ $=\left(\frac{6}{25}\right)\left[\frac{\left(\frac{2}{5}\right)^3}{1-\frac{2}{5}}+\frac{\left(\frac{3}{5}\right)^3}{1-\frac{3}{5}}\right]$ $=0.155$ Or P(first obtaining beads of different colours after 5 or more draws) = P(obtaining same colour in the first 4 draws) = P(first 4 red beads) + P(first 4 green beads) $=\left(\frac{2}{5}\right)^4+\left(\frac{3}{5}\right)^4=0.155$
9	Assume that the mass of sodium in a box follows a normal distribution. To test $H_0 : \mu = 1183$ $H_1 : \mu \neq 1183$ at 5% level of significance Under H_0, $T=\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t(14)$ Critical Region: Reject H_0 if p-value < 0.05 Calculation: $\bar{x}=\frac{\sum(x-1180)}{15}+1180=\frac{17.5}{15}+1180=1181.166667$, $s^2=\frac{1}{n-1}\left(\sum(x-1180)^2-\frac{\left(\sum(x-1180)\right)^2}{n}\right)$, $=\frac{1}{14}\left(190.5-\frac{17.5^2}{15}\right)=12.14880952$ n=15 $t_{calc}=\frac{1181.166667-1183}{\left(\sqrt{\frac{12.14880952}{15}}\right)}$

	$p-\text{value}=0.0610$ Conclusion: Since p-value = 0.0610 > 0.05, we do not reject H_0. There is insufficient evidence, at 5% level of significance, to say that the mean mass of sodium in a box of cheese rings has changed. (ii) To test $H_0 : \mu = \mu_0$ $H_1 : \mu > \mu_0$ at 5% level of significance Under H_0, $Z=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \text{N}(0,1)$ Critical Region: Reject H_0 if $z_{calc} > 1.64485$ Calculation: $\bar{x}=1181.166667$, $\sigma^2=6.8^2$, n=15 $z_{cal}=\frac{\bar{x}-\mu_0}{6.8/\sqrt{15}}$ Since H_0 is rejected, $z_{cal}>1.64485$, ie. $\mu_0<1178.28$ The assumption of the masses being normal distribution is still necessary as the sample size is small/Cannot use CLT
10	(i) Let X be the number of minutes after 6:30 am that Peter takes to reach the bus stop. $X \sim N(0,10^2)$ Prob Req'd = $P(X>20)=0.0228$ (ii) Let Y be the number of minutes for the bus journey. $Y \sim N(45,20^2)$ Prob.Req′d $= P(X>20)+P(X\le 10)\cdot P(Y>50)+P(10<X\le 20)\cdot P(Y>40)$ $\approx 0.44174 \approx 0.442$ Let W be the number of days in a five days week in which Peter will be late for school. $W \sim B(5,0.442)$. Prob. Req′d $= P(W\ge 2)$ $=1-P(W\le 1)$ $=0.732$

	$E(W) = 2.21,\ Var(W) = 1.23318$ By Central Limit Theorem, $\overline{W} \sim N(2.21, 0.0308295)$ approx. $\text{Prob. Req'd} = P(\overline{W} \geq 2)$ ≈ 0.884
11	Let X be the number of calls reporting on a lost card in 1 hour, $X \sim \text{Po}(18)$ (i)Required prob = $\text{P}(X > 12) = 1 - \text{P}(X \leq 12)$ = 0.908(3308) (ii)Let Y be the total number of calls received in 15 minutes, $Y \sim \text{Po}(4.5+3+6)$ Required prob = $\text{P}(Y \leq 12) = 0.409$ (iii)Let W be the number of one-hour periods where more than 12 calls reporting a lost card are received in an hour, out of 24 one-hour periods. $W \sim \text{B}(24, \text{part I answer})$ Required prob = $\text{P}(W \geq 20) = 1 - \text{P}(W \leq 19)$ = 0.937 binomial, 0.936947, sensitive to i) answer used in 3 dp; 0.9361448 if 0.908 is used (iv)Let U be number of nuisance calls received in 1 hour, and V be number of calls requesting for increase in credit limit in 1 hour. $U \sim \text{Po}(24)$, $V \sim \text{Po}(12)$, and $X \sim \text{Po}(18)$ (from part i)) can be approximated by normal distributions $U \sim \text{N}(24, 24)$, $V \sim \text{N}(12,12)$ and $X \sim \text{N}(18,18)$ respectively since the Poisson means are more than 10. Hence, $U - (X+V)$ approximately $\text{N}(24 - (12+18), 24+12+18)$ i.e. $\text{N}(-6, 54)$ Required prob = $\text{P}(U > X + V) = \text{P}(U - (X+V)) > 0)$ = 0.188 (with c.c.) or 0.207 (without c.c.) (v)Assumptions Number of phone calls received of the various types are independent Phone calls of the various types are mutually exclusive Not accepted – enough telephone operators to receive the calls (because the poisson distn is already on calls "received")