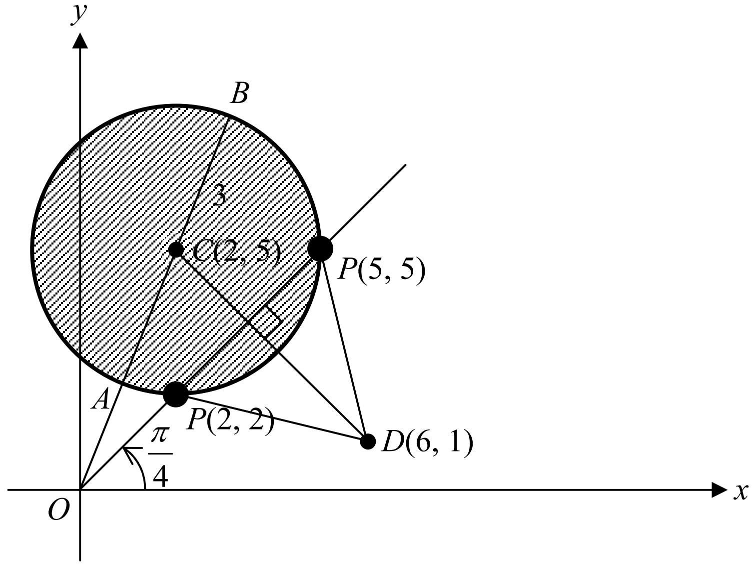
Nov 2011 H2 Maths Paper 2 Solutions

1. (i) $|z-(2+5\text{i})| \le 3$



(ii) $OC = \sqrt{2^2+5^2} = \sqrt{29}$

Maximum value of $|z| = OB = OC + CB = \sqrt{29}+3$.

Minimum value of $|z| = OA = OC - CA = \sqrt{29}-3$.

(iii) Maximum value of $|z-6-\text{i}| = DP = \sqrt{(2-6)^2+(2-1)^2} = \sqrt{17}$.

2. (i) $V = PQ \times QR \times \text{height}$

$$\begin{aligned} &= (2n-2x)(n-2x)x \\ &= x\left(2n^2-4nx-2nx+4x^2\right) \\ &= 2n^2x-6nx^2+4x^3 \text{ (shown)} \end{aligned}$$

(ii) $\dfrac{\mathrm{d}V}{\mathrm{d}x} = 2n^2-12nx+12x^2$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 0 \Rightarrow 6x^2-6nx+n^2=0$$

$$\Rightarrow x = \frac{6n \pm \sqrt{36n^2-4(6)n^2}}{12} = \frac{6n \pm \sqrt{12n^2}}{12} = \frac{6n \pm 2\sqrt{3}n}{12} = \left(\frac{3\pm\sqrt{3}}{6}\right)n$$

Since $2x<n$, $x<\dfrac{n}{2}$ $\Rightarrow x = \left(\dfrac{3-\sqrt{3}}{6}\right)n$ $\left[\left(\dfrac{3+\sqrt{3}}{6}\right)n \text{ rej} \because \left(\dfrac{3+\sqrt{3}}{6}\right)n > \dfrac{n}{2}\right]$

$\therefore$ Stationary value of V occurs only when $x = \left(\dfrac{3-\sqrt{3}}{6}\right)n$.

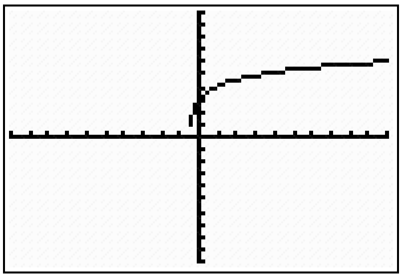
Note: No need to carry out 1st/2nd derivative test because the question only asks for the value of *x* that gives a <u>stationary</u> value of *V* (not maximum/minimum)

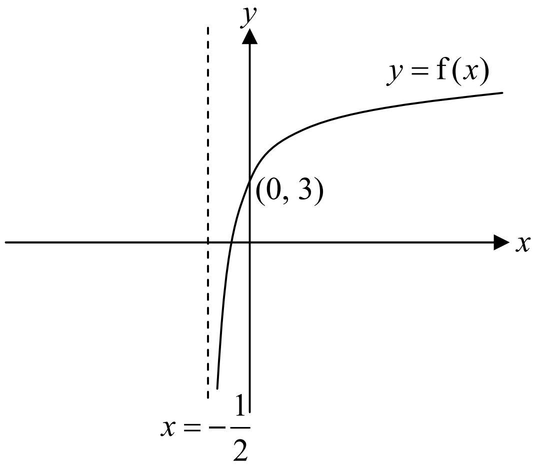
3. (i) Let $y=\ln(2x+1)+3$.

$\Rightarrow \ln(2x+1)=y-3$

$\Rightarrow 2x+1=\mathrm{e}^{y-3}$

$\Rightarrow x=\dfrac{\mathrm{e}^{y-3}-1}{2}$



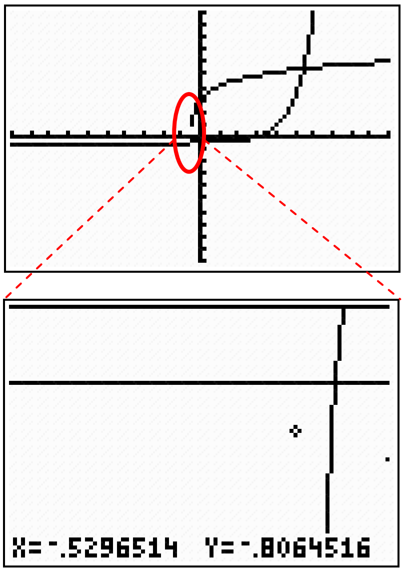


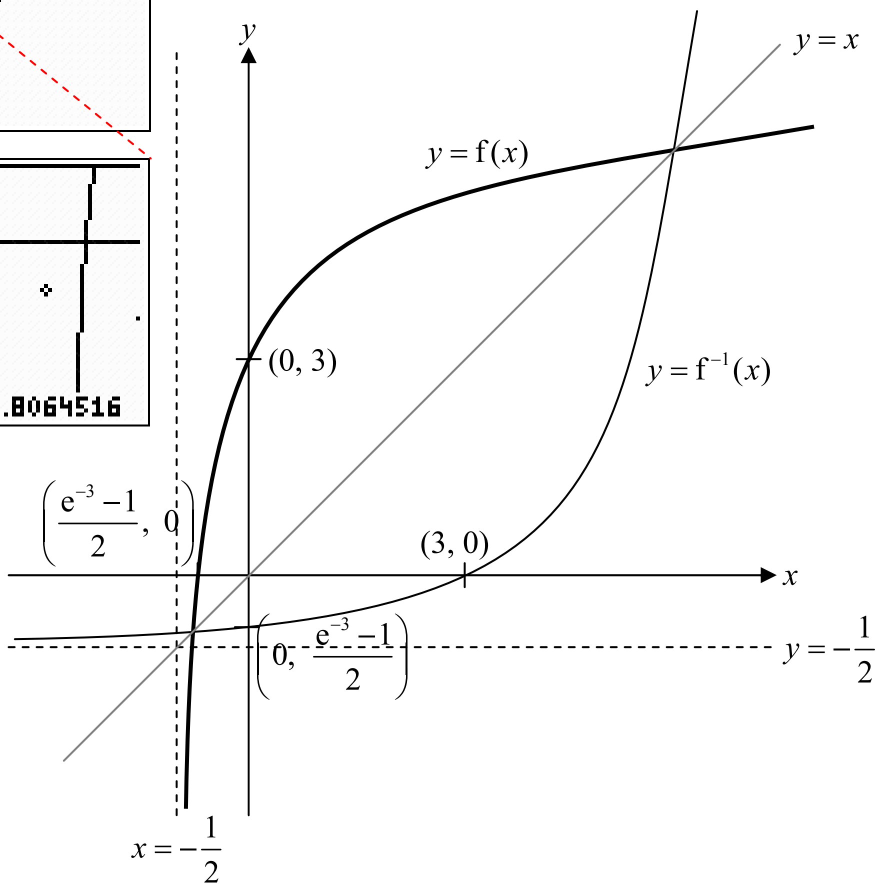
$D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\mathbb{R}$

$\therefore\quad \mathrm{f}^{-1}(x)=\dfrac{\mathrm{e}^{x-3}-1}{2},\ x\in\mathbb{R}.$

$R_{\mathrm{f}^{-1}}=D_{\mathrm{f}}=\left(-\dfrac{1}{2},\ \infty\right)$

(ii)

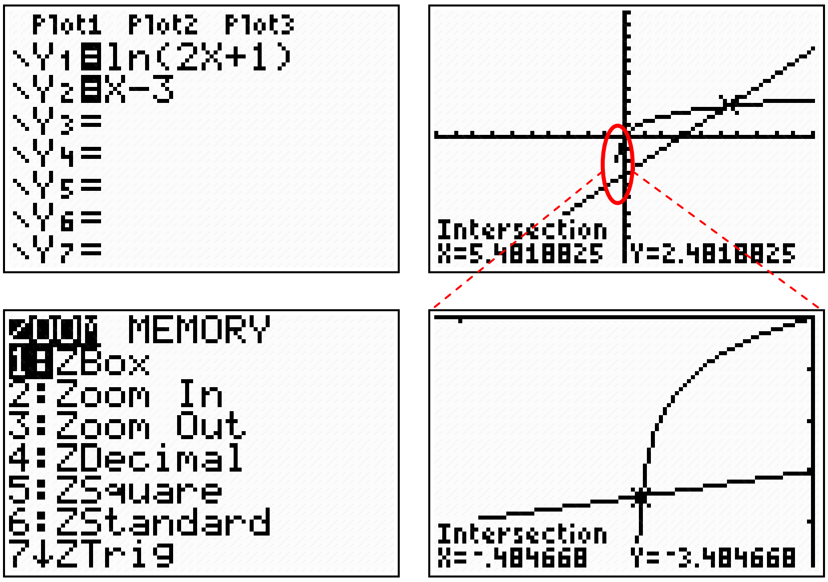




(iii) The points of intersection of the two curves $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ lie on the line $y = x$. Therefore, the x-coordinates of the points of intersection satisfy the equation $\mathrm{f}(x) = x$, i.e.

$\Rightarrow \ln(2x+1) + 3 = x$

$\Rightarrow \ln(2x+1) = x - 3$.



Using GC, $x = -0.4847$ or 5.482 (to 4 sf).

4. (a) (i) $\int_0^n x^2 e^{-2x}\ dx = \left[x^2\left(\frac{e^{-2x}}{-2}\right)\right]_0^n - \int_0^n \left(\frac{e^{-2x}}{-2}\right)2x\ dx$

$$= -\frac{n^2 e^{-2n}}{2} + \int_0^n x e^{-2x}\ dx$$

$$= -\frac{n^2 e^{-2n}}{2} + \left[x\left(\frac{e^{-2x}}{-2}\right)\right]_0^n - \int_0^n \frac{e^{-2x}}{-2}\ dx$$

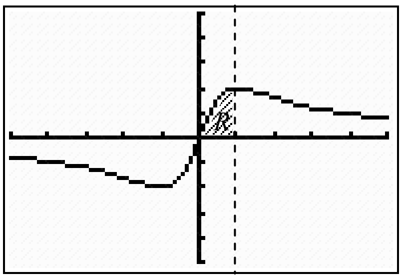
$$= -\frac{n^2 e^{-2n}}{2} - \frac{n e^{-2n}}{2} + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right]_0^n$$

$$= -\frac{n^2 e^{-2n}}{2} - \frac{n e^{-2n}}{2} - \frac{1}{4}\left(e^{-2n} - 1\right)$$

$$= \frac{1}{4} - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{4}\right)e^{-2n}$$

(ii) $\int_0^\infty x^2 e^{-2x}\ dx = \lim_{n\to\infty}\left[\frac{1}{4} - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{4}\right)e^{-2n}\right] = \frac{1}{4}$

(b)



Volume of solid $= \pi\int_0^1 \left(\frac{4x}{x^2+1}\right)^2\ dx$

Let $x = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta$.

$x = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Volume of solid $= \pi\int_0^1 \left(\frac{4x}{x^2+1}\right)^2\ dx$

$$= 16\pi\int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{(\tan^2\theta+1)^2}(\sec^2\theta)\ d\theta$$

$$= 16\pi\int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\sec^2\theta}\ d\theta$$

$$= 16\pi\int_0^{\frac{\pi}{4}} \sin^2\theta\ d\theta \text{ (shown)}$$

$$= 8\pi\int_0^{\frac{\pi}{4}} \left(1-\cos 2\theta\right)\, \mathrm{d}\theta$$

$$= 8\pi\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= 8\pi\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= 2\pi^2 - 4\pi$$

$$= 2\pi(\pi - 2)$$

5. $X \sim \mathrm{N}(\mu,\ \sigma^2)$

$\mathrm{P}(X < 40.0) = 0.05 \quad \Rightarrow \mathrm{P}\left(Z < \dfrac{40-\mu}{\sigma}\right) = 0.05$

$$\Rightarrow \frac{40-\mu}{\sigma} = -1.6449$$

$$\Rightarrow \mu - 1.6449\sigma = 40 \quad \text{----------(1)}$$

$\mathrm{P}(X < 70.0) = 0.975 \quad \Rightarrow \mathrm{P}\left(Z < \dfrac{70-\mu}{\sigma}\right) = 0.975$

$$\Rightarrow \frac{70-\mu}{\sigma} = 1.9600$$

$$\Rightarrow \mu + 1.9600\sigma = 70 \quad \text{----------(2)}$$

Using GC, $\mu = 53.7$ and $\sigma = 8.32$.

6. (i) Stratify the residents of the city suburb into different age groups.

Stand on a street corner and ask people who walk past until the required number of people in each age group have been interviewed.

(ii) The sample obtained is non-random as not every resident in the city suburb has an equal chance of being selected. For example, the interviewer might choose people who he thinks are more approachable.

(iii) Stratified sampling.

It would not be realistic to use this method as it might be difficult to obtain a list of all the residents of the city suburb (i.e. the sampling frame).

7. (i) The two assumptions are:
   (1) The n attempts are independent of each other.
   (2) The probability of successfully contacting is the same for each attempt.

(ii) The second assumption mentioned in part (i) may not hold as the probability of successfully contacting varies from friend to friend.

(iii) $R \sim \mathrm{B}(8,\ 0.7)$

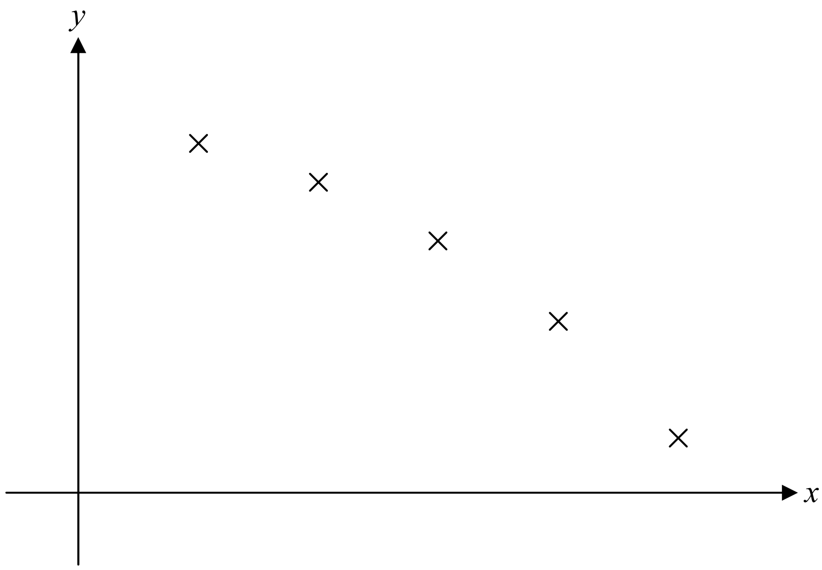
$\mathrm{P}(R \geq 6) = 1 - \mathrm{P}(R \leq 5) \approx 0.55177 = 0.552$ (to 3 sig. fig.)

(iv) $R \sim \mathrm{B}(40,\ 0.7)$

Since $n = 40$ is sufficiently large, $np = 40(0.7) = 28 > 5$ and $nq = 40(0.3) = 12 > 5$, $R \sim \mathrm{N}(28,\ 8.4)$ approximately.

$\mathrm{P}(R < 25) = \mathrm{P}(R \leq 24) \xrightarrow{c.c.} \mathrm{P}(R < 24.5) \approx 0.11360 = 0.114$ (to 3 sig. fig.)

8. (i)



(ii) $r = -0.99232 \approx -0.992$.

The product moment correlation coefficient measures the strength and direction of a linear relationship between two variables. It is possible for the relationship to be non-linear and yet having r close to -1. We also need to look at the scatter diagram in order to determine whether the relationship is linear.

(iii) For $y = a + bx^2$, $r = -0.99998$.
For $y = c + dx$, $r = -0.99232$.

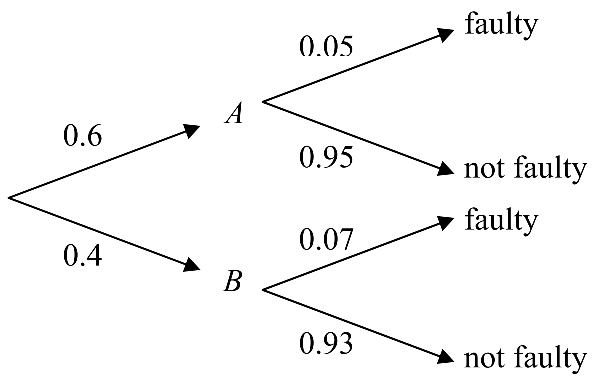
Between the two values of r, $r = -0.99998$ is closer to -1. Therefore, $y = a + bx^2$ is the better model.

(iv) $y = 22.230 - 0.85621x^2$

$y = 22.2 - 0.856x^2$

When $x = 3.2$, $y = 22.230 - 0.85621(3.2)^2 = 13.462 \approx 13.5$ (to 3 sig. fig.)

9.



(i) (a) P(faulty) $=$ P(made by A and faulty) $+$ P(made by B and faulty)

$= 0.6(0.05) + 0.4(0.07)$

$= 0.058$

(b) $$\text{P(made by } A \mid \text{faulty)} = \frac{\text{P(made by } A \text{ and faulty)}}{\text{P(faulty)}}$$

$$= \frac{0.6(0.05)}{0.058}$$

$$= \frac{15}{29} \text{ or } 0.517$$

(ii) (a) P(exactly one of them is faulty) $= 0.058 \times (1 - 0.058) \times 2!$

$= 0.109272$

(b) P(both were made by A | exactly one is faulty)

$$= \frac{\text{P(both were made by } A \text{ and exactly one is faulty)}}{\text{P(exactly one is faulty)}}$$

$$= \frac{\text{P(one is made by } A \text{ and faulty, the other is made by } A \text{ and not faulty)}}{\text{P(exactly one is faulty)}}$$

$$= \frac{[0.6(0.05)] \times [0.6(0.95)] \times 2!}{0.109272}$$

$$= \frac{1425}{4553} \text{ or } 0.313$$

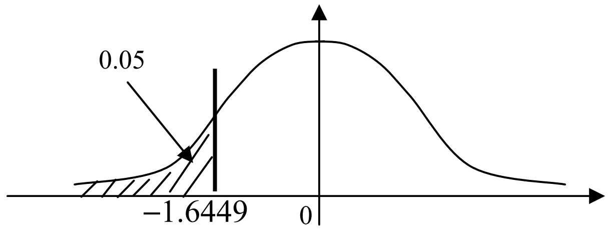
10. (i) Null hypothesis $\mathrm{H_0}: \mu = 38$

Alternative hypothesis $\mathrm{H_1}: \mu < 38$

where μ is the population mean time taken to install an electronic component.

(ii) Under $\mathrm{H_0}$, $\overline{T} \sim \mathrm{N}\left(38, \frac{5^2}{n}\right)$.

Hence, test statistic $Z = \dfrac{\overline{T} - 38}{\sqrt{5^2/n}} \sim \mathrm{N}(0,\ 1)$.



For $\alpha = 0.05$, reject $\mathrm{H_0}$ if $z \le -1.6449$.

$\Rightarrow \dfrac{\overline{t} - 38}{\sqrt{5^2/50}} \le -1.6449$

$\Rightarrow \overline{t} \le 36.8$

(iii) For $\alpha = 0.05$, do not reject $\mathrm{H_0}$ if $z > -1.6449$.

$\Rightarrow \dfrac{37.1 - 38}{\sqrt{5^2/n}} > -1.6449$

$\Rightarrow -0.18\sqrt{n} > -1.6449$

$\Rightarrow \sqrt{n} < \dfrac{1.6449}{0.18}$

$\Rightarrow n < 83.5$

$\Rightarrow n \le 83$

11. (i) $\mathrm{P}(R = 4) = \dfrac{{}^{18}C_4\,{}^{12}C_6}{{}^{30}C_{10}} = \dfrac{816}{8671}$ or 0.0941

(ii) $\mathrm{P}(R = r) > \mathrm{P}(R = r+1)$

$\Rightarrow \dfrac{{}^{18}C_r\,{}^{12}C_{10-r}}{{}^{30}C_{10}} > \dfrac{{}^{18}C_{r+1}\,{}^{12}C_{9-r}}{{}^{30}C_{10}}$

$\Rightarrow \left(\dfrac{18!}{r!(18-r)!}\right)\left(\dfrac{12!}{(10-r)!(2+r)!}\right) > \left(\dfrac{18!}{(r+1)!(17-r)!}\right)\left(\dfrac{12!}{(9-r)!(3+r)!}\right)$

$\Rightarrow (r+1)!(17-r)!(9-r)!(r+3)! > r!(18-r)!(10-r)!(r+2)!$ (shown)

By simplifying the factorial terms,

$(r+1)!(17-r)!(9-r)!(r+3)! > r!(18-r)!(10-r)!(r+2)!$

$\Rightarrow \dfrac{(r+1)!}{r!}\dfrac{(r+3)!}{(r+2)!} > \dfrac{(18-r)!}{(17-r)!}\dfrac{(10-r)!}{(9-r)!}$

$\Rightarrow (r+1)(r+3) > (18-r)(10-r)$

$\Rightarrow r^2+4r+3 > 180-28r+r^2$

$\Rightarrow 32r > 177$

$\Rightarrow r > 5.53$

$\therefore \quad r = 6$.

12(i) Let X = no. of people joining the queue.
In 4 minutes, $X \sim \mathrm{Po}(1.2\times 4) = \mathrm{Po}(4.8)$.

$\mathrm{P}(X \geq 8) = 1-\mathrm{P}(X \leq 7) \approx 0.11333 = 0.113$ (to 3 sig. fig.)

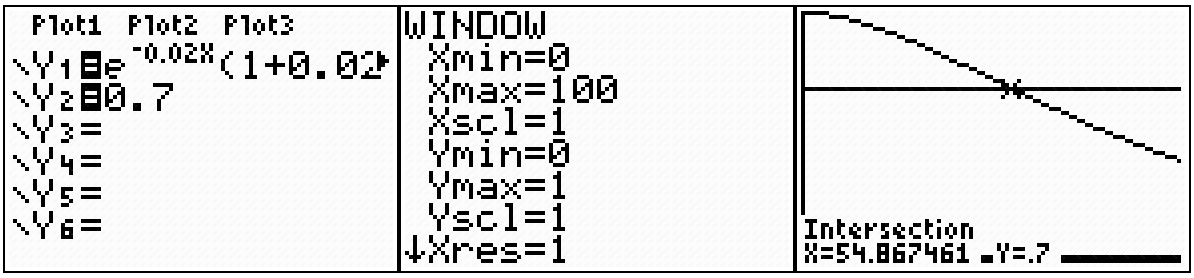
(ii) In t seconds, $X \sim \mathrm{Po}\left(1.2\times\dfrac{t}{60}\right) = \mathrm{Po}(0.02t)$

$\mathrm{P}(X \leq 1) = 0.7$

$\Rightarrow \mathrm{P}(X=0)+\mathrm{P}(X=1) = 0.7$

$\Rightarrow \dfrac{\mathrm{e}^{-0.02t}\left(0.02t\right)^0}{0!}+\dfrac{\mathrm{e}^{-0.02t}\left(0.02t\right)^1}{1!} = 0.7$

$\Rightarrow \mathrm{e}^{-0.02t}(1+0.02t) = 0.7$



Using GC, $t = 55$ (to nearest whole number).

(iii) In 15 minutes, $X \sim \mathrm{Po}(1.2\times 15) = \mathrm{Po}(18)$.
Since $\lambda = 18 > 10$, $X \sim \mathrm{N}(18,\ 18)$ approximately.

Let Y = no. of people leaving the queue.
In 15 minutes, $Y \sim \mathrm{Po}(1.8\times 15) = \mathrm{Po}(27)$.
Since $\lambda = 27 > 10$, $Y \sim \mathrm{N}(27,\ 27)$ approximately.

Therefore, $X - Y \sim \mathrm{N}(-9,\ 45)$ approximately.

$$
\begin{aligned}
\mathrm{P}(35+X-Y\geq 24) &= \mathrm{P}(X-Y\geq -11)\\
&\xrightarrow{c.c.}\mathrm{P}(X-Y>-11.5)\\
&\approx 0.64531\\
&= 0.645\left(\text{to 3 sig. fig.}\right)
\end{aligned}
$$

(iv) The average number of people joining the queue per minute may not remain constant throughout a time period of several hours possibly due to people travelling in groups or different departure time. Thus a Poisson model would probably not be valid.