Nov 2011 H2 Maths Paper 1 Solutions

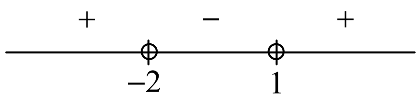
1. $\dfrac{x^2+x+1}{x^2+x-2}<0$

$$\Rightarrow \frac{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}{(x-1)(x+2)}<0$$

$$\Rightarrow \frac{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}{(x-1)(x+2)}<0$$

Since $\left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0$ for all x,

$\Rightarrow (x-1)(x+2)<0$



$\therefore \quad -2<x<1$

2. (i) When $x=-1.5$, $y=4.5$:

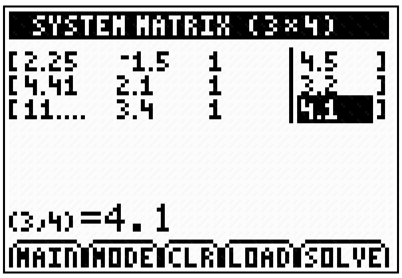
$(-1.5)^2a+(-1.5)b+c=4.5 \quad \Rightarrow 2.25a-1.5b+c=4.5$ ----------(1)

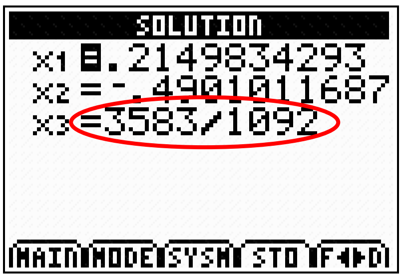
When $x=2.1$, $y=3.2$:

$(2.1)^2a+(2.1)b+c=3.2 \quad \Rightarrow 4.41a+2.1b+c=3.2$ ----------(2)

When $x=3.4$, $y=4.1$:

$(3.4)^2a+(3.4)b+c=4.1 \quad \Rightarrow 11.56a+3.4b+c=4.1$ ----------(3)





Using GC, $a=0.215$, $b=-0.490$, $c=3.281$ (all to 3 dp).

(ii) For $f(x)$ to be an increasing function, $f'(x)>0$.

$\Rightarrow 2ax+b>0$

$\Rightarrow x>\dfrac{0.49010}{2\times 0.21498}=1.14$

$\therefore \quad x>1.14$

3. (i) $\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = -\frac{2}{t^2}$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{2}{t^2}}{2t} = -\frac{1}{t^3}$

At the point $\left(p^2, \ \frac{2}{p}\right)$, $t = p$.

When $t = p$, $\frac{dy}{dx} = -\frac{1}{p^3}$.

Hence, equation of the tangent at the point $\left(p^2, \ \frac{2}{p}\right)$ is

$y - \frac{2}{p} = -\frac{1}{p^3}(x - p^2)$

$\therefore \quad y = -\frac{x}{p^3} + \frac{3}{p}$

(ii) At the x-axis, $y = 0 \Rightarrow x = 3p^2$

$\therefore \quad Q$ is $(3p^2, \ 0)$.

At the y-axis, $x = 0 \Rightarrow y = \frac{3}{p}$

$\therefore \quad R$ is $\left(0, \ \frac{3}{p}\right)$.

(iii) Mid-point of QR is $\left(\frac{3p^2 + 0}{2}, \ \frac{0 + \frac{3}{p}}{2}\right) = \left(\frac{3p^2}{2}, \ \frac{3}{2p}\right)$.

$x = \frac{3p^2}{2}$ ----------(1)

$y = \frac{3}{2p}$ ----------(2)

From (2), $p = \frac{3}{2y}$ ----------(3)

Substitute (3) into (1):

$x = \frac{3\left(\frac{3}{2y}\right)^2}{2} = \frac{\frac{27}{4y^2}}{2} = \frac{27}{8y^2}$

$8xy^2 = 27$

A cartesian equation of the locus of the mid-point of QR is $8xy^2 = 27$.

4. (i) $g(x) = \cos^6 x$

$$= (\cos x)^6$$

$$\approx \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)^6$$

$$= 1^6 + \binom{6}{1}(1)^5\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^1 + \binom{6}{2}(1)^4\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + \dots$$

$$= 1 + 6\left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + 15\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + \dots$$

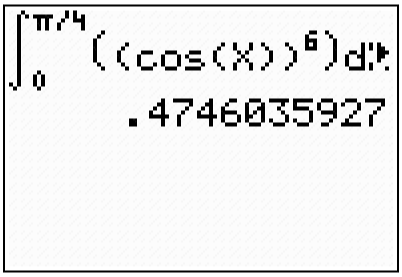
$$= 1 - 3x^2 + \frac{1}{4}x^4 + 15\left(\frac{x^4}{4}\right) + \dots$$

$$\approx 1 - 3x^2 + 4x^4$$

(ii) (a) $\int_0^a g(x)\, dx \approx \int_0^a (1 - 3x^2 + 4x^4)\, dx = \left[x - x^3 + \frac{4x^5}{5}\right]_0^a = a - a^3 + \frac{4a^5}{5}$

When $a = \frac{\pi}{4}$, $\int_0^a g(x)\, dx \approx \frac{\pi}{4} - \left(\frac{\pi}{4}\right)^3 + \frac{4}{5}\left(\frac{\pi}{4}\right)^5 = 0.540$ (to 3 sf).

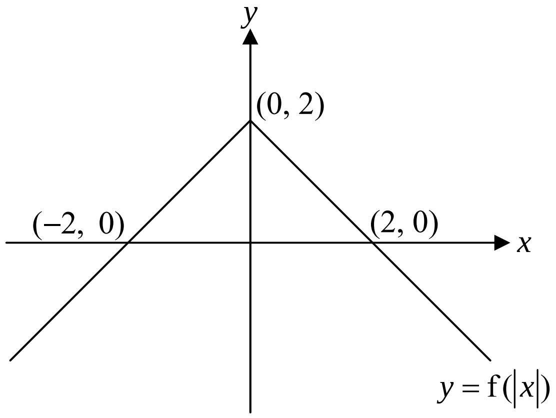
(b)

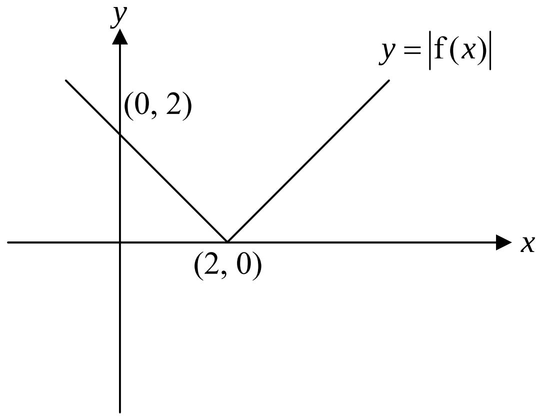


Using GC, $\int_0^{\frac{\pi}{4}} g(x)\, dx = 0.475$ (to 3 sf).

The approximation in part (ii) (a) is not very good as $a = \frac{\pi}{4}$ is not very close to 0.

5. (i)





(ii) $0 \le x \le 2$

(iii) <u>Method 1: Using integration</u>

$$\int_{-1}^{1} \mathrm{f}(|x|)\ \mathrm{d}x = \int_{1}^{a} |\mathrm{f}(x)|\ \mathrm{d}x$$

$$\Rightarrow 2\int_{0}^{1} (2-x)\ \mathrm{d}x = \int_{1}^{2} (2-x)\ \mathrm{d}x + \int_{2}^{a} (x-2)\ \mathrm{d}x$$

$$\Rightarrow 2\left[2x - \frac{x^2}{2}\right]_0^1 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^a$$

$$\Rightarrow 3 = \frac{1}{2} + \left(\frac{a^2}{2} - 2a + 2\right)$$

$$\Rightarrow a^2 - 4a - 1 = 0$$

$$\Rightarrow a = \frac{4 \pm \sqrt{20}}{2} = 2 + \sqrt{5} \quad \left(2 - \sqrt{5} \text{ rej } \because a > 2\right)$$

<u>Method 2: Using areas of trapezia and triangles</u>

$$\int_{-1}^{1} \mathrm{f}(|x|)\ \mathrm{d}x = \int_{1}^{a} |\mathrm{f}(x)|\ \mathrm{d}x$$

$$\Rightarrow 2 \times \frac{1}{2}(1)(2+1) = \frac{1}{2}(1)(1) + \frac{1}{2}(a-2)(a-2)$$

$$\Rightarrow (a-2)^2 = 5$$

$$\Rightarrow a - 2 = \pm\sqrt{5}$$

$$\Rightarrow a = 2 + \sqrt{5} \quad \left(2 - \sqrt{5} \text{ rej } \because a > 2\right)$$

6. (i) $\sin\left(r+\frac{1}{2}\right)\theta-\sin\left(r-\frac{1}{2}\right)\theta$

$$=\left(\sin r\theta\cos\frac{\theta}{2}+\cos r\theta\sin\frac{\theta}{2}\right)-\left(\sin r\theta\cos\frac{\theta}{2}-\cos r\theta\sin\frac{\theta}{2}\right)$$

$$=2\cos r\theta\sin\frac{\theta}{2} \text{ (proven)}$$

(ii) $$\sum_{r=1}^{n}\cos r\theta=\sum_{r=1}^{n}\frac{\left[\sin\left(r+\frac{1}{2}\right)\theta-\sin\left(r-\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}$$

$$=\frac{1}{2\sin\frac{\theta}{2}}\sum_{r=1}^{n}\left[\sin\left(r+\frac{1}{2}\right)\theta-\sin\left(r-\frac{1}{2}\right)\theta\right]$$

$$=\frac{1}{2\sin\frac{\theta}{2}}\Bigg[\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}$$
$$+\sin\frac{5\theta}{2}-\sin\frac{3\theta}{2}$$
$$+\ldots$$
$$+\sin\left(n+\frac{1}{2}\right)\theta-\sin\left(n-\frac{1}{2}\right)\theta\Bigg]$$

$$=\frac{\sin\left(n+\frac{1}{2}\right)\theta-\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

(iii) Let P_n be the statement $\sum_{r=1}^{n} \sin r\theta = \dfrac{\cos\frac{\theta}{2} - \cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$ for all $n \in \mathbb{Z}^+$.

LHS of $P_1 = \sin\theta$

$$\text{RHS of } P_1 = \frac{\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{-\left(\cos\frac{3\theta}{2} - \cos\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}} = \frac{-\left(-2\sin\theta\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}} = \sin\theta$$

$\therefore$ P_1 is true.

Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e. $\sum_{r=1}^{k} \sin r\theta = \dfrac{\cos\frac{\theta}{2} - \cos\left(k+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$.

We want to prove P_{k+1}, i.e. $\sum_{r=1}^{k+1} \sin r\theta = \dfrac{\cos\frac{\theta}{2} - \cos\left(k+\frac{3}{2}\right)\theta}{2\sin\frac{\theta}{2}}$.

$$\begin{aligned}
\text{LHS of } P_{k+1} &= \sum_{r=1}^{k+1} \sin r\theta \\
&= \sum_{r=1}^{k} \sin r\theta + \sin(k+1)\theta \\
&= \frac{\cos\frac{\theta}{2} - \cos\left(k+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} + \sin(k+1)\theta \\
&= \frac{\cos\frac{\theta}{2} - \cos\left(k+\frac{1}{2}\right)\theta + 2\sin(k+1)\theta\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \\
&= \frac{\cos\frac{\theta}{2} - \cos\left(k+\frac{1}{2}\right)\theta - \left[\cos\left(k+\frac{3}{2}\right)\theta - \cos\left(k+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}} \\
&= \frac{\cos\frac{\theta}{2} - \cos\left(k+\frac{3}{2}\right)\theta}{2\sin\frac{\theta}{2}} = \text{RHS of } P_{k+1}
\end{aligned}$$

$\therefore$ P_{k+1} is true.

Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

7. (i) $\overrightarrow{OP}=\frac{1}{3}\overrightarrow{OA}=\frac{1}{3}\mathbf{a}$, $\overrightarrow{OQ}=\frac{3}{5}\overrightarrow{OB}=\frac{3}{5}\mathbf{b}$.

Using Ratio Theorem, $\overrightarrow{OM}=\frac{\overrightarrow{OP}+\overrightarrow{OQ}}{2}=\frac{1}{2}\left(\frac{1}{3}\mathbf{a}+\frac{3}{5}\mathbf{b}\right)=\frac{1}{6}\mathbf{a}+\frac{3}{10}\mathbf{b}$.

Area of triangle *OMP*

$=\frac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OM}\right|$

$=\frac{1}{2}\left|\frac{1}{3}\mathbf{a}\times\left(\frac{1}{6}\mathbf{a}+\frac{3}{10}\mathbf{b}\right)\right|$

$=\frac{1}{2}\left|\left(\frac{1}{18}\mathbf{a}\times\mathbf{a}\right)+\left(\frac{1}{10}\mathbf{a}\times\mathbf{b}\right)\right|$

$=\frac{1}{2}\left|\mathbf{0}+\left(\frac{1}{10}\mathbf{a}\times\mathbf{b}\right)\right|$ since $\mathbf{a}\times\mathbf{a}=\mathbf{0}$

$=\frac{1}{20}\left|\mathbf{a}\times\mathbf{b}\right|$

(ii) (a) Since **a** is a unit vector, $|\mathbf{a}|=1$.

$\Rightarrow\sqrt{4p^2+36p^2+9p^2}=1$

$\Rightarrow 7p=1$

$\Rightarrow p=\frac{1}{7}$

(b) $|\mathbf{a}.\mathbf{b}|=|\mathbf{b}.\mathbf{a}|$ is the length of projection of **b** on **a**.

(c) $\mathbf{a}\times\mathbf{b}=\frac{1}{7}\begin{pmatrix}2\\-6\\3\end{pmatrix}\times\begin{pmatrix}1\\1\\-2\end{pmatrix}=\frac{1}{7}\begin{pmatrix}9\\7\\8\end{pmatrix}$

8. (i) $\displaystyle\int \frac{1}{100-v^2}\,\mathrm{d}v = \frac{1}{20}\ln\left|\frac{10+v}{10-v}\right| + c$

(ii) (a) $\displaystyle\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.1v^2$

$$\Rightarrow \int \frac{1}{10-0.1v^2}\,\mathrm{d}v = \int \mathrm{d}t$$

$$\Rightarrow \int \frac{1}{100-v^2}\,\mathrm{d}v = \int \frac{1}{10}\,\mathrm{d}t$$

$$\Rightarrow \frac{1}{20}\ln\left|\frac{10+v}{10-v}\right| = \frac{1}{10}t + c$$

When $t = 0$, $v = 0$: $c = 0$.

$$\therefore \quad t = \frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|.$$

When $v = 5$, $t = \frac{1}{2}\ln 3$.

(b) $\displaystyle t = \frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|$

$$\Rightarrow \left|\frac{10+v}{10-v}\right| = \mathrm{e}^{2t}$$

$$\Rightarrow \frac{10+v}{10-v} = \pm\mathrm{e}^{2t} = \mathrm{e}^{2t} \quad \text{(select positive root} \because v = 0 \text{ when } t = 0\text{)}$$

$$\Rightarrow 10 + v = \mathrm{e}^{2t}(10 - v)$$

$$\Rightarrow v(1+\mathrm{e}^{2t}) = 10(\mathrm{e}^{2t} - 1)$$

$$\Rightarrow v = \frac{10(\mathrm{e}^{2t}-1)}{\mathrm{e}^{2t}+1}$$

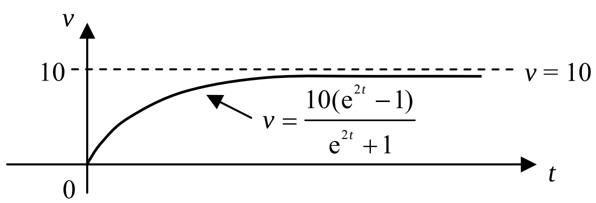
When $t = 1$, $v = \dfrac{10(\mathrm{e}^2 - 1)}{\mathrm{e}^2 + 1} = 7.62$ (to 3 sf)

(c) <u>Method 1: Evaluate limits</u>

$$v = \frac{10(\mathrm{e}^{2t}-1)}{\mathrm{e}^{2t}+1} = \frac{10(1-\mathrm{e}^{-2t})}{1+\mathrm{e}^{-2t}}$$

As $t \to \infty$, $v \to 10$.

<u>Method 2: Sketch graph</u>



As $t \to \infty$, $v \to 10$.

9. (i) Machine A: Depth drilled each day follows an AP with $a = 256$ and $d = -7$

Depth drilled on the 10th day $= T_{10} = 256 + 9(-7) = 193$ metres.

$T_n < 10$
$\Rightarrow 256 + (n-1)(-7) < 10$
$\Rightarrow 253 < 7n$
$\Rightarrow n > 36.1$

Total depth when drilling is completed $= S_{37}$

$$= \frac{37}{2}[2(256) + 36(-7)]$$
$$= 4810 \text{ metres.}$$

(ii) Machine B: Depth drilled each day follows a GP with $a = 256$ and $r = \frac{8}{9}$

$S_n > 0.99 S_\infty$

$$\Rightarrow \frac{a(1-r^n)}{1-r} > 0.99\left(\frac{a}{1-r}\right)$$

$\Rightarrow 1 - r^n > 0.99$

$$\Rightarrow \left(\frac{8}{9}\right)^n < 0.01$$

$$\Rightarrow n > \frac{\ln 0.01}{\ln \frac{8}{9}} = 39.1$$

It takes 40 days.

10. (i) $z^2 = -8\mathrm{i} = 8\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} = 8\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}+2k\pi\right)}$

$$\Rightarrow z = \sqrt{8}\mathrm{e}^{\mathrm{i}\left(\frac{-\frac{\pi}{2}+2k\pi}{2}\right)},\ k = 0,\ 1$$

$$= 2\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}},\ 2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$$

$$= 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right],\ 2\sqrt{2}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)$$

$$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}}-\mathrm{i}\frac{1}{\sqrt{2}}\right),\ 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\mathrm{i}\frac{1}{\sqrt{2}}\right)$$

$$= 2-2\mathrm{i},\ -2+2\mathrm{i}$$

(ii) $w^2 + 4w + (4+2\mathrm{i}) = 0$

$$w = \frac{-4 \pm \sqrt{16-4(4+2\mathrm{i})}}{2}$$

$$= \frac{-4 \pm \sqrt{-8\mathrm{i}}}{2}$$

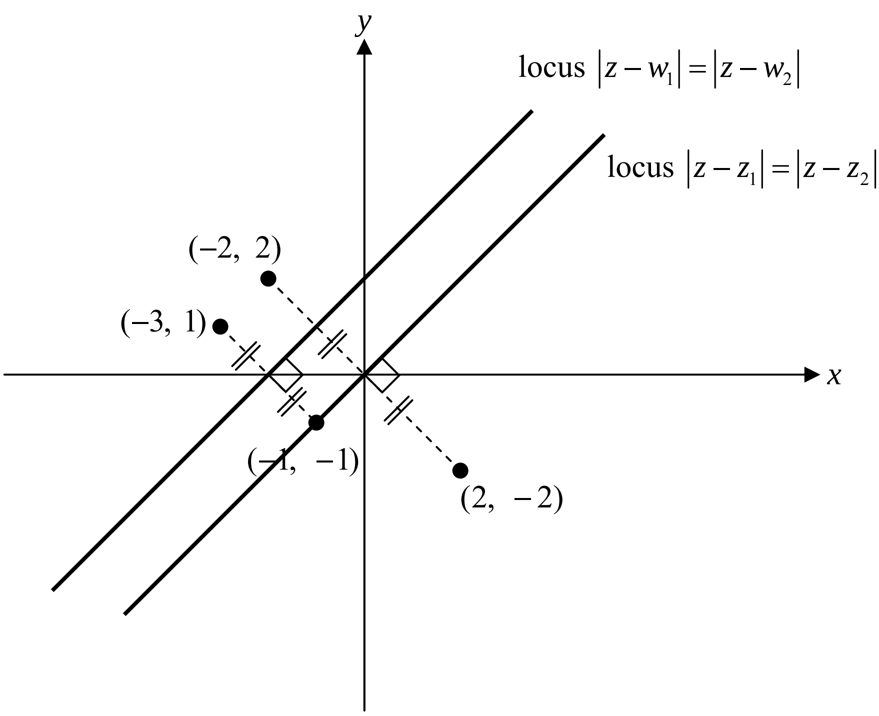
$$= \frac{-4 \pm \sqrt{z^2}}{2}$$

$$= \frac{-4 \pm z}{2}$$

$$= \frac{-4+(2-2\mathrm{i})}{2},\ \frac{-4+(-2+2\mathrm{i})}{2}$$

$$= -1-\mathrm{i},\ -3+\mathrm{i}$$

(iii)



(iv) Since the two perpendicular bisectors are parallel to each other, there are no points which lie on both loci.

11. (i) Two vectors parallel to plane p are $\begin{pmatrix}-2\\-5\\2\end{pmatrix}-\begin{pmatrix}4\\-1\\-3\end{pmatrix}=\begin{pmatrix}-6\\-4\\5\end{pmatrix}$ and $\begin{pmatrix}4\\-3\\-2\end{pmatrix}-\begin{pmatrix}4\\-1\\-3\end{pmatrix}=\begin{pmatrix}0\\-2\\1\end{pmatrix}$.

$$\mathbf{n}=\begin{pmatrix}-6\\-4\\5\end{pmatrix}\times\begin{pmatrix}0\\-2\\1\end{pmatrix}=\begin{pmatrix}6\\6\\12\end{pmatrix}=6\begin{pmatrix}1\\1\\2\end{pmatrix}$$

Vector equation of p is $\mathbf{r.}\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}4\\-1\\-3\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix} \quad \Rightarrow \mathbf{r.}\begin{pmatrix}1\\1\\2\end{pmatrix}=-3$.

A cartesian equation of p is $x+y+2z=-3$.

(ii) $l_1:\ \mathbf{r}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\1\end{pmatrix}, \quad l_2:\ \mathbf{r}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\5\\k\end{pmatrix}$.

$$\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\5\\k\end{pmatrix}$$

$$\Rightarrow\begin{cases}2\lambda-\mu = -3 & ----------(1)\\ -4\lambda-5\mu = -1 & ----------(2)\\ \lambda-k\mu = 6 & ----------(3)\end{cases}$$

Using GC to solve (1) and (2), $\lambda=-1$ and $\mu=1$.
Substitute into (3): $-1-k=6\Rightarrow k=-7$.

(iii) <u>To show l_1 lies in p: Method 1</u>

Since $\begin{pmatrix}2\\-4\\1\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix}=0$, l_1 is parallel to p.

Since $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix}=-3$, $(1,\ 2,\ -3)$ lies in p.

$\therefore$ l_1 lies in p (shown).

To show l_1 lies in p: Method 2

Since $\left[\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=(1+2\lambda)+(2-4\lambda)+2(-3+\lambda)=-3$, l_1 lies in p.

$$\left[\begin{pmatrix}-2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\5\\-7\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=-3$$

$\Rightarrow 5-8\mu=-3$

$\Rightarrow \mu=1$

$$\begin{pmatrix}-2\\1\\3\end{pmatrix}+\begin{pmatrix}1\\5\\-7\end{pmatrix}=\begin{pmatrix}-1\\6\\-4\end{pmatrix}$$

$\therefore$ l_2 intersects p at $(-1,\ 6,\ -4)$.

(iv) Acute angle between l_2 and $p=\sin^{-1}\dfrac{\left|\begin{pmatrix}1\\5\\-7\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}\right|}{\sqrt{75}\sqrt{6}}=\sin^{-1}\dfrac{8}{\sqrt{75}\sqrt{6}}=22.2°$ (to 1 dp)