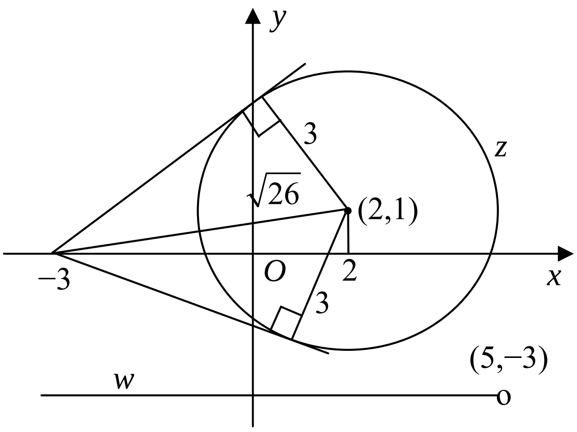
Anglo-Chinese Junior College

H2 Mathematics 9740

2010 JC 2 PRELIM Marking Scheme

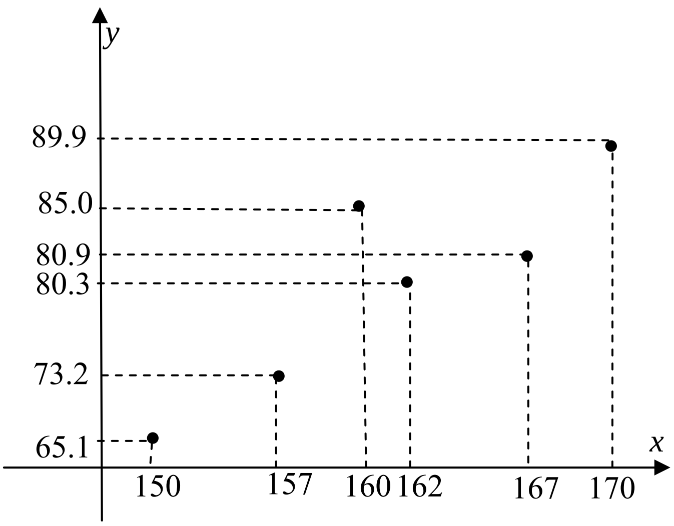
Paper 2:

1	$\int_{-1}^{1} \left\| e^{2x} - \frac{1}{e^{2(x-1)}} \right\| \, dx$ $= -\int_{-1}^{1/2} e^{2x} - \frac{1}{e^{2(x-1)}} \, dx + \int_{1/2}^{1} e^{2x} - \frac{1}{e^{2(x-1)}} \, dx$ $= -\left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2(x-1)}\right]_{-1}^{1/2} + \left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2(x-1)}\right]_{1/2}^{1}$ $= \frac{1}{2}\left(e^4 + e^2 - 4e + e^{-2} + 1\right)$
2 (i)	
2 (ii)	Least value of $\|z-w\| = 1$
2 (iii)	Greatest $\arg(z+3) = \tan^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\left(\frac{3}{\sqrt{26}}\right) = 0.826$ (3 dp) Least $\arg(z+3) = \tan^{-1}\left(\frac{1}{5}\right) - \sin^{-1}\left(\frac{3}{\sqrt{26}}\right) = -0.432$ (3 dp)
3	$\ln\left(2x - \frac{a}{x}\right) \geq 0$ where $a > 1$. $2x - \frac{a}{x} \geq 1$ $\frac{2x^2 - x - a}{x} \geq 0$ $\frac{2\left(x - \frac{1+\sqrt{1+8a)}}{4}\right)\left(x - \frac{1-\sqrt{1+8a)}}{4}\right)}{x} \geq 0$ $\frac{1-\sqrt{1+8a}}{4} \leq x < 0$ *or* $x \geq \frac{1+\sqrt{1+8a}}{4}$
4 (a)	$\frac{d}{dx}\left(\cos x^3\right) = -3x^2\left(\sin x^3\right)$

4 (b)	$\int x^5 \sin x^3\, dx$ $= \int x^3 (x^2 \sin x^3)\, dx$ $= x^3\left(-\frac{1}{3}\cos x^3\right) - \int\left(-\frac{1}{3}\cos x^3\right)3x^2\, dx$ $= -\frac{x^3}{3}\cos x^3 + \int x^2 \cos x^3\, dx$ $= -\frac{x^3}{3}\cos x^3 + \frac{1}{3}\sin x^3 + c$
	$x = \sqrt{5}\sec\theta$ $\quad$ *when* $x = \sqrt{10},\ \theta = \frac{\pi}{4}$ $\frac{dx}{d\theta} = \sqrt{5}\sec\theta\tan\theta$ $\quad$ *when* $x = 2\sqrt{5},\ \theta = \frac{\pi}{3}$ $\int_{\sqrt{10}}^{2\sqrt{5}} \left(x^2 - 5\right)^{-\frac{3}{2}} dx$ $= \int_{\pi/4}^{\pi/3} \left(5\sec^2\theta - 5\right)^{-\frac{3}{2}} \sqrt{5}\sec\theta\tan\theta\, d\theta$ $= \frac{1}{5}\int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan^2\theta} d\theta$ $= \frac{1}{5}\int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta} d\theta$ *or* $\frac{1}{5}\int_{\pi/4}^{\pi/3} \cot\theta\, \mathrm{cosec}\,\theta\, d\theta$ $= \frac{1}{5}\left[-\frac{1}{\sin\theta}\right]_{\pi/4}^{\pi/3}$ $\quad = \frac{1}{5}\left[-\mathrm{cosec}\,\theta\right]_{\pi/4}^{\pi/3}$ $= \frac{1}{5}\sqrt{2} - \frac{2}{15}\sqrt{3}$ $\quad$ *i.e.* $a = \frac{1}{5}$ $\quad b = -\frac{2}{15}$
5 (i)	$p_1 : \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 3$ $\qquad$ $l_1 : \underset{\sim}{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix},\ \lambda \in \mathbb{R}$ $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 2 + 2 - 4 = 0$ $\qquad$ $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -1 + 0 - 1 \neq 3$ $\Rightarrow$ l_1 is parallel to p_1. $\qquad$ $\Rightarrow$ l_1 is not contained in p_1. ***Alternative method:*** $\begin{pmatrix} -1+2\lambda \\ \lambda \\ 1+4\lambda \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -2 \neq 3$ Since no solution for λ, $\therefore$ l_1 is parallel and not contained to p_1
(ii)	$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = -3\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

	$p_2 : \underset{\sim}{r} \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = 3 + 0 + 1 = 4 \qquad p_2 : -3x + 2y + z = 4$
(iii)	$p_3 : \underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 8 + 1 - 4 = 5 \qquad p_3 : \underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 5$
(iv)	$l_2 : \underset{\sim}{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix},\ \mu \in \mathbb{R} \qquad p_1 : \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 3$ $\begin{pmatrix} -1 + 2\mu \\ 0 \\ 1 - 3\mu \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 3 \Rightarrow -2 + 5\mu = 3 \Rightarrow \mu = 1$ $\therefore$ point B is $(1, 0, -2)$
(v)	$\sin\theta = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \frac{5}{\sqrt{78}}$ Length of the projection of AB on p_1 = $\left\| \overrightarrow{AB} \right\| \cos\theta$ $= \left\| \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \right\| \sqrt{1 - \frac{25}{78}} = \sqrt{13}\left(\sqrt{\frac{53}{78}}\right) = \sqrt{\frac{53}{6}} = \frac{1}{6}\sqrt{318}$
6	Procedure A is preferable. Early customers may not be typical customers in general.
7	Let R be the r.v for the length of a right shoe and L for the left shoe R $\sim$ N(20, 0.14^2) and L $\sim$ N(20.1 , 0.11^2) **<u>Method 1</u>** X = R + L $\sim$ N(40.1, 0.0317) 3X $\sim$ N(40.1 x 3, 0.0317 x 3^2) P(X > 120) = 0.713 **<u>Method 2</u>** R + L $\sim$ N(40.1, 0.0317) P(R + L > $\frac{120}{3}$) = 0.713
8	Let M and W be the rv for the weight of an adult man and woman respectively. M $\sim$ N($75,4^2$) and W $\sim$ N($65,3^2$) W − M ~ N(-10, 5^2) P($\lvert W - M \rvert > 1$) = P(W – M > 1) + P(W – M< -1) = 0.978 Or P($\lvert W - M \rvert > 1$) = 1 - P($\lvert W - M \rvert < 1$) = 1 – P(-1<W – M<1) = 0.978

	No , (i) The weight of a husband and wife may not be independent Or (ii) Randomness is not there *(a randomly chosen women but spouse is not randomly chosen)* Or (iii) Distribution of weight of married woman is different from distribution of adult woman. Etc
9	Telephone costs are assumed to be normally distributed. To test H_0 : $\mu = 72$ against H_1: $\mu > 72$ at 5% level of significance Under H_0 , $\frac{\bar{x}-\mu_0}{s/\sqrt{n}} \sim t(5\text{-}1)$ Test statistics : $T= \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{82.6-72}{10.6677/\sqrt{5}} = 2.2219$ p value = P(T > 2.2219) = 0.0452 < 0.05 **Reject** H_0 at the 5% level of significance. We conclude that there is sufficient **evidence** at the **5% level of significance** that there is evidence of an increase in mean monthly costs. To test H_0 : $\mu = \mu_0$ against H_1: $\mu > \mu_0$ at 5% level of significance Under H_0 , $\frac{\bar{x}-\mu_0}{9.89/\sqrt{n}} \sim N(0,1)$ Test statistics : $Z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} = \frac{82.6-\mu_0}{9.89/\sqrt{5}} = (82.6 - \mu_0)\frac{\sqrt{5}}{9.89}$ Do not reject H_0 if $P(Z > (82.6 - \mu_0)\frac{\sqrt{5}}{9.89}) > 0.05$ $(82.6 - \mu_0)\frac{\sqrt{5}}{9.89} < 1.64485$...........(1) $\mu_0 > 75.3$
10 (i)	$\bar{x} = 161$ (from calculator or computation)

	when $\bar{x}=161$, $\bar{x}=103.6+0.726\bar{y}$ $\bar{y}=(161-103.6)/0.726$ $=79.06336088$ using $\bar{y}=\sum y/n$ $79.06336088=\frac{1}{6}(65.1+73.2+85+k+80.9+89.9)$ $k=80.3$ Use G.C. to find regression line of *y* on *x*: $y=-97.593+1.097x$
(ii)	Use *y* on *x* line to predict weight. When $x=165$, $y=-97.593+1.097(165)$ $y=83.4$ (1 d.p.) – using 3 d.p. of *a* and *b* to compute. or $y=83.5$ - using full accuracy of *a* and *b* to compute.
(iii)	Using G.C., $r=0.893$  C is unusually overweight.
11	Breakages occur randomly or Breakages occur independently or **Mean** number of breakages is a constant
(i)	Let A be the r.v for the number of broken cups per day A ~Po(2.1) P(A ≥ 3) = 1 – P(A ≤ 2) = 0.350369 = 0.350 (3 sig figs) Let X be the r.v for the number of days with a least 3 broken cups X~$B(n, 0.350)$

$P(X \geq 2) > 0.999$

$1 - P(X \leq 1) > 0.999$............(1)

$P(X \leq 1) < 0.001$

$n \leq 22$

least n is 22

(ii)

$T \sim Po(2.1x7 + 1.6 \text{ x } 7)$

$T \sim Po(25.9)$

<u>Method 1:</u>

Since n is large,

$\bar{T} \sim N(25.9, \frac{25.9}{100})$ approx by central limit theorem

$P(\bar{T} \leq 26) = 0.578$ (to 3 sig fig)

<u>Method 2:</u>

100 weeks, $Y \sim Po(2590)$

$\lambda = 2590 > 10$. Normal approx to Poisson

$Y \sim N(2590,2590)$ approx

$P(Y \leq 2600) = P(Y < 2600.5)$ (With cc)

= 0.578 (to 3 sig fig)

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(i)

Let X be the r.v for the number of fish which measures less than 8 cm long.

${}^{9}C_{5}(\frac{1}{5})^{5}(\frac{4}{5})^{4}(\frac{1}{5}) = 0.00330$ (to 3 sf)

(ii)

$X \sim B(n,0.2)$

Since n large and p= 0.2 , $X \sim N(0.2n,(0.2)(0.8)n)$ approx

$P(X \leq 10) \leq 0.0227$

$P(X < 10.5) \leq 0.0227$

$P(Z < \frac{10.5 - 0.2n}{0.4\sqrt{n}}) \leq 0.0227$

$\frac{10.5 - 0.2n}{0.4\sqrt{n}} \leq -2.000929$..........(1)

$10.5 - 0.2n \leq -0.800372\sqrt{n}$

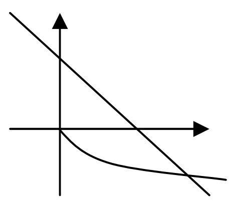
Hence $10.5 - 0.2n \leq -0.8\sqrt{n}$ approx

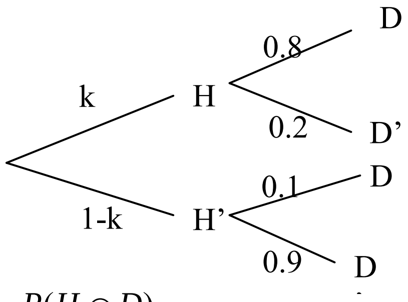
Method 1

Using GC $Y_1 = 10.5 - 0.2x + 0.8\sqrt{x}$

→Table Ans : 91

<u>Method 2</u> : Use GC and graph



	<u>Method 3</u>: Hence $(10.5 – 0.2n)^2 \geq (-0.8\sqrt{n})^2$ Hence $4n^2 - 484n + 11025 \geq 0$ approx……(2) From GC : $n \geq 90.6$ or $n \leq 30.4$ $n \geq 91$ or $n \leq 30$ (NA because does not satisfy (1)) Least n = 91
13 **(i)** **(ii)**	Let H be the event that the member of public has hypertension Let D be the event that the machine diagnosed hypertension  (i) P(H/D) = $\frac{2}{3} = \frac{P(H \cap D)}{P(D)}$ $\frac{2}{3} = \frac{0.8k}{0.8k + (1-k)(0.1)}$ (1) k = 0.2 p% = 20 % P(H'/D') = $\frac{P(H' \cap D')}{P(D')} = \frac{(1-k)(0.9)}{0.2k + (1-k)(0.9)} = 0.947$ The machine is a good tool for initial screening. If it does not find you hypertensive then you can be reasonably confident that your blood pressure is normal. If it diagnoses hypertension, then you should consult your doctor for further tests.
14 **(i)** **(ii)**	**Case 1:** 2 red balls - $\binom{2}{2} = 1$ **Case 2:** 2 blue balls - $\binom{3}{2} = 3$ **Case 3:** 2 green balls - $\binom{5}{2} = 10$ No. of ways = $1 + 3 + 10 = 14$ **Case 1: No red ball.** Choose 6 balls from a total of 8 (blue and green) balls: $\binom{8}{6}$. **Case 2: No blue ball**.

	Choose 6 balls rom a total of 7 (red and green) balls: $\binom{7}{6}$ [Note: We can't exclude green balls because total number of red and blue balls is only 5.] No. of ways = $\binom{10}{6}-\left[\binom{8}{6}+\binom{7}{6}\right]=210-(28+7)=175$

Alternative Method:

Case	Green	Blue	Red	No. of ways
1	4	1	1	$\binom{5}{4}\times\binom{3}{1}\times\binom{2}{1}=30$
2	3	2	1	$\binom{5}{3}\times\binom{3}{2}\times\binom{2}{1}=60$
3	3	1	2	$\binom{5}{3}\times\binom{3}{1}\times\binom{2}{2}=30$
4	2	3	1	$\binom{5}{3}\times\binom{3}{2}\times\binom{2}{1}=60$
5	2	2	2	$\binom{5}{2}\times\binom{3}{2}\times\binom{2}{2}=20$
6	1	3	2	$\binom{5}{1}\times\binom{3}{3}\times\binom{2}{2}=5$
			Total	175

(iii)

Excluding the green balls, we only have 1, 1, 2, 2, 3. Since we are ignoring the colours of the balls, we are forming 3-digit numbers from the 5 digits 1, 1, 2, 2, 3.

Case 1: All 3 digits are distinct.

The 3 digits are 1, 2, 3 and the number of ways of arranging them are 3!

Case 2: 2 digits are identical.

Step 1: Choose 2 digits that are identical (1, 1 or 2, 2): 2

Step 2: Choose a digit from the remaining digits (1, 3 or 2, 3): 2

Step 3: Arrange the 3 chosen digits in a row:

$$\frac{3!}{2!}=3$$

No. of ways = $3!+\left(\frac{3!}{2!}\right)(2)(2)=6+12=18$