

Anglo-Chinese Junior College
H2 Mathematics 9740
2008 JC 2 PRELIM PAPER 1 Solutions

1	$\frac{2-x}{\sqrt{1+3x}} = (2-x)(3x)^{-\frac{1}{2}} \left(1 + \frac{1}{3x}\right)^{-\frac{1}{2}}$ $= \frac{1}{\sqrt{3}} \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) \left(1 + \frac{1}{3x}\right)^{-\frac{1}{2}}$ $= \frac{1}{\sqrt{3}} \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) \left(1 - \frac{1}{6x} + \frac{1}{24x^2} - \dots\right)$ $= \frac{1}{\sqrt{3}} \left(-x^{\frac{1}{2}} + \frac{13}{6}x^{\frac{1}{2}} - \frac{3}{8}x^{\frac{3}{2}}\right)$ <p>range of validity: $x > \frac{1}{3}$ or $x < -\frac{1}{3}$</p> <p style="text-align: center;">but since $x < -\frac{1}{3}$ will result in the sq rt of a negative number, reject $x < -\frac{1}{3}$.</p> <p style="text-align: center;">\therefore range of validity: $x > \frac{1}{3}$.</p>	
2	<p>no. of terms in the sum: $n + 2$</p> $\text{sum} = \frac{n+2}{2} \left[\left(-\frac{1}{2} + (m-1)d\right) + \left(-\frac{1}{2} + (m+n)d\right) \right]$ $= \frac{n+2}{2} [-1 + (2m+n-1)d]$	
3	<p>Method 1</p> $\int e^x (\cos^2 x - 1) dx = \int e^x \left(\frac{\cos 2x + 1}{2} - 1 \right) dx$ $= \frac{1}{2} \int e^x \cos 2x - e^x dx$ <div style="display: flex; justify-content: space-between;"> <div style="width: 45%;"> $I = \int e^x \cos 2x dx$ $= \frac{1}{2} e^x \sin 2x - \frac{1}{2} \int e^x \sin 2x dx$ $= \frac{1}{2} e^x \sin 2x - \frac{1}{2} \left(-\frac{1}{2} e^x \cos 2x + \frac{1}{2} \int e^x \cos 2x dx \right)$ $= \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4} I$ $\frac{5}{4} I = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x$ $I = \frac{2}{5} e^x \sin 2x + \frac{1}{5} e^x \cos 2x$ </div> <div style="width: 45%;"> $u_1 = e^x, \frac{dv_1}{dx} = \cos 2x$ $\frac{du_1}{dx} = e^x, v_1 = \frac{1}{2} \sin 2x$ $u_2 = e^x, \frac{dv_2}{dx} = \sin 2x$ $\frac{du_2}{dx} = e^x, v_2 = -\frac{1}{2} \cos 2x$ </div> </div> <p>Thus,</p> $\int e^x (\cos^2 x - 1) dx = \frac{1}{2} \left[\frac{2}{5} e^x \sin 2x + \frac{1}{5} e^x \cos 2x \right] - \frac{1}{2} e^x + C$ $= \frac{1}{5} e^x \sin 2x + \frac{1}{10} e^x \cos 2x - \frac{1}{2} e^x + C$	

Method 2

$$\int e^x (\cos^2 x - 1) dx = \int e^x \cos^2 x dx - \int e^x dx$$

$$\int e^x \cos^2 x dx$$

$$\frac{dv_1}{dx} = e^x, u_1 = \cos^2 x$$

$$v_1 = e^x, \frac{du_1}{dx} = -\sin 2x$$

$$= -e^x \cos^2 x + \int e^x \sin 2x dx$$

$$u_2 = e^x, \frac{dv_2}{dx} = \sin 2x$$

$$\frac{du_2}{dx} = e^x, v_2 = -\frac{1}{2} \cos 2x$$

$$= -e^x \cos^2 x - \frac{1}{2} e^x \cos 2x + \frac{1}{2} \int e^x \cos 2x dx$$

$$u_3 = e^x, \frac{dv_3}{dx} = \cos 2x$$

$$\frac{du_3}{dx} = e^x, v_3 = \frac{1}{2} \sin 2x$$

$$= -e^x \cos^2 x - \frac{1}{2} e^x \cos 2x + \frac{1}{2} \left(\frac{1}{2} e^x \sin 2x - \frac{1}{2} \int e^x \sin 2x dx \right)$$

$$= -e^x \cos^2 x - \frac{1}{2} e^x \cos 2x + \frac{1}{2} e^x \sin 2x - \frac{1}{4} \int e^x \sin 2x dx$$

$$\text{Let } I = \int e^x \sin 2x dx$$

Then,

$$-e^x \cos^2 x + I = -e^x \cos^2 x - \frac{1}{2} e^x \cos 2x + \frac{1}{2} e^x \sin 2x - \frac{1}{4} I$$

$$\frac{5}{4} I = \frac{1}{4} e^x \sin 2x - \frac{1}{2} e^x \cos 2x$$

$$I = \frac{1}{5} e^x \sin 2x - \frac{2}{5} e^x \cos 2x$$

$$\text{Thus, } \int e^x (\cos^2 x - 1) dx = \frac{1}{5} e^x \sin 2x - \frac{2}{5} e^x \cos 2x - e^x + C$$

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$$x: 10, -\frac{5}{2}, \frac{5}{8}, -\frac{5}{32}, \dots$$

$$y: 5, -\frac{5}{4}, \frac{5}{16}, -\frac{5}{64}, \dots$$

$$x: \text{GP with } a = 10 \text{ and } r = -\frac{1}{4}$$

$$y: \text{GP with } a = 5 \text{ and } r = -\frac{1}{4}$$

$$x: S_{\infty} = \frac{10}{1 + \frac{1}{4}} = 8, \quad y: S_{\infty} = \frac{5}{1 + \frac{1}{4}} = 4$$

the ant will eventually end up at (8, 4).

<p>5</p>	<p>For $x-2 \geq 0 \Rightarrow x \geq 2$,</p> $x(x-2) \geq 1$ $x^2 - 2x - 1 \geq 0$ $(x-1)^2 - 2 \geq 0$ $(x-1-\sqrt{2})(x-1+\sqrt{2}) \geq 0$ $x \leq 1-\sqrt{2} \text{ (N.A.) or } x \geq 1+\sqrt{2}$	<p>ALTERNATIVE:</p> <p>For $x-2 < 0 \Rightarrow x < 2$,</p> $x(2-x) \geq 1$ $x^2 - 2x + 1 \leq 0$ $(x-1)^2 \leq 0$ <p>No real solution except $x = 1$</p>	
<p>Hence, $x \geq 1+\sqrt{2}$, $x = 1$</p>			
	<p>$(x+1) x-1 \geq 1$</p> <p>Let $u = x + 1$, $u u-2 \leq 1$ B1</p> <p>$\therefore u = x+1 \geq 1+\sqrt{2} \Rightarrow x \geq \sqrt{2}$,</p> <p>$u = x+1 = 1 \Rightarrow x = 0$</p> <p style="text-align: right;">B2</p>	<p>OTHERWISE</p> <p>sketch of $y = (x+1) x-1 - 1$</p> <p>hence, $x \geq \sqrt{2}$, $x = 0$</p>	
<p>6</p>	<p>$x = \ln(\cos \theta)$, $y = \ln(\sin \theta)$</p> $\frac{dx}{d\theta} = -\frac{\sin \theta}{\cos \theta}$ $\frac{dy}{d\theta} = \frac{\cos \theta}{\sin \theta}$ $\frac{dy}{dx} = \frac{\cos \theta}{\sin \theta} \left(-\frac{\cos \theta}{\sin \theta} \right)$ $= -\frac{\cos^2 \theta}{\sin^2 \theta}$ <p>when $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = -1$</p> $\Rightarrow \frac{y - \ln \frac{1}{\sqrt{2}}}{x - \ln \frac{1}{\sqrt{2}}} = -1$ <p>The equation of tangent is $y = -x - \ln 2$.</p> <p>If this tangent meets the curve again,</p> $\ln(\sin \theta) = -\ln(\cos \theta) - \ln 2$ $\ln(\sin 2\theta) = 0$ $\sin 2\theta = 1$ $\therefore 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4} \quad \left(0 < \theta < \frac{\pi}{2} \right)$ <p>The tangent will not meet the curve again as there is only one solution in range $0 < \theta < \frac{\pi}{2}$ i.e. $\theta = \frac{\pi}{4}$.</p>		

7 By Newton's Law of Cooling,

$$\frac{d\theta}{dt} = k(\theta - 20)$$

$$\int \frac{d\theta}{\theta - 20} = \int k dt$$

$$\therefore \ln|\theta - 20| = kt + C \text{ --- ①}$$

Method 1

Given: when $t = t_1$, $\theta = 60$ — **A**

When $t = t_1 + 10$, $\theta = 45$ — **B**

From ①: $\ln 40 = kt_1 + C$, $\ln 25 = kt_1 + 10k + C$

$$\Rightarrow \ln \frac{25}{40} = 10k$$

$$\Rightarrow k = \frac{1}{10} \ln \frac{5}{8} \text{ --- ②}$$

Also given: $t = t_2$, $\theta = ?$

$t = t_2 + 20$, $\theta = 45$ — **C**

From ① & ②: $\ln|\theta - 20| = \left(\frac{1}{10} \ln \frac{5}{8}\right)t_2 + C_1$,

$$\ln 25 = \left(\frac{1}{10} \ln \frac{5}{8}\right)t_2 + 20\left(\frac{1}{10} \ln \frac{5}{8}\right) + C_1$$

$$\Rightarrow \ln \frac{25}{\theta - 20} = 2 \ln \frac{5}{8} \quad [\text{Modulus can be removed as initial temperature is higher than } 45^\circ\text{C, hence } \theta - 20 > 0.]$$

$$\Rightarrow \theta = \frac{25}{\left(\frac{5}{8}\right)^2} + 20 = 84^\circ\text{C}$$

Method 2

Given: $t = 0$, $\theta = 60$ — **A**

$t = 10$, $\theta = 45$ — **B**

From ①: $\ln 40 = C$, $\ln 25 = 10k + C$

$$\Rightarrow \ln \frac{25}{40} = 10k$$

$$\Rightarrow k = \frac{1}{10} \ln \frac{5}{8} \text{ --- ②}$$

Also given: $t = 0$, $\theta = ?$

$t = 20$, $\theta = 45$ — **C**

From ① & ②: $\ln|\theta - 20| = \left(\frac{1}{10} \ln \frac{5}{8}\right)t_2 + C_1$,

$$\ln 25 = 20\left(\frac{1}{10} \ln \frac{5}{8}\right) + C_1$$

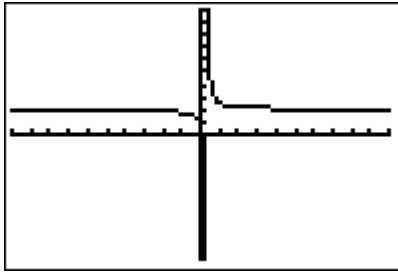
$$\Rightarrow \ln \frac{25}{\theta - 20} = 2 \ln \frac{5}{8} \quad [\text{Modulus can be removed as initial temperature is higher than } 45^\circ\text{C, hence } \theta - 20 > 0.]$$

$$\Rightarrow \frac{25}{\theta - 20} = \left(\frac{5}{8}\right)^2$$

$$\Rightarrow \theta = \frac{25}{\left(\frac{5}{8}\right)^2} + 20 = 84^\circ\text{C}$$

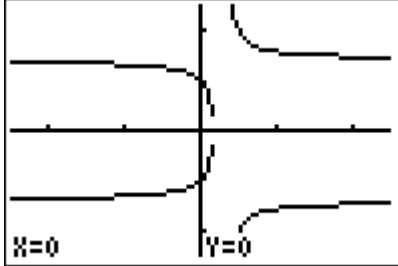
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$$a = 6, \quad b = -1$$



Asymptotes $y = 2, \quad x = \frac{1}{3}$

Axial intersection : $(0,1), (1/6,0)$



Asymptotes $y = \pm\sqrt{2}, \quad x = \frac{1}{3}$

Axial intersection : $(0,1), (0,-1)$

$$y = \frac{6x-1}{3x-1} = 2 + \frac{1}{3x-1}$$

I : A translation of 1 unit in direction of the positive x – axis

II : A scaling parallel to the x – axis with scale factor $\frac{1}{3}$ units

III: A translation of 2 unit in direction of the positive y – axis

OR

I : A translation of 2 unit in direction of the positive y – axis

II : A translation of 1 unit in direction of the positive x – axis

III : A scaling parallel to the x – axis with scale factor $\frac{1}{3}$ units

OR

I : A translation of 1 unit in direction of the positive x – axis

II : A translation of 2 unit in direction of the positive y – axis

III: A scaling parallel to the x – axis with scale factor $\frac{1}{3}$ units

OR

I : A scaling parallel to the x – axis with scale factor $\frac{1}{3}$ units

II : A translation of $\frac{1}{3}$ unit in direction of the positive x – axis

III: A translation of 2 unit in direction of the positive y – axis

OR

I : A scaling parallel to the x – axis with scale factor $\frac{1}{3}$ units

II : A translation of 2 unit in direction of the positive y – axis

III: A translation of $\frac{1}{3}$ unit in direction of the positive x – axis

OR

I : A translation of 2 unit in direction of the positive y – axis

II: A scaling parallel to the x – axis with scale factor $\frac{1}{3}$ units

III :A translation of $\frac{1}{3}$ unit in direction of the positive x – axis

9

$$y = \sin^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

$$\Rightarrow \sqrt{1-x^2} \frac{dy}{dx} = 1$$

diff. w.r.t x ,

$$\sqrt{1-x^2} \frac{d^2y}{dx^2} + \frac{dy}{dx} \left(\frac{-2x}{2\sqrt{1-x^2}} \right) = 0$$

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 0$$

diff. w.r.t x ,

$$(1-x^2) \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} (-2x) - \left(x \frac{d^2y}{dx^2} + \frac{dy}{dx} \right) = 0$$

$$(1-x^2) \frac{d^3y}{dx^3} - 3x \frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \quad (QED)$$

diff. w.r.t x ,

$$(1-x^2) \frac{d^4y}{dx^4} - 5x \frac{d^3y}{dx^3} - 4 \frac{d^2y}{dx^2} = 0$$

diff. w.r.t x ,

$$(1-x^2) \frac{d^5y}{dx^5} - 7x \frac{d^4y}{dx^4} - 9 \frac{d^3y}{dx^3} = 0$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = 1, \quad f^{(4)}(0) = 0, \quad f^{(5)}(0) = 9$$

$$y = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{d}{dx} \sin^{-1} x$$

$$= \frac{d}{dx} \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots \right)$$

$$= 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \dots$$

$$\text{Using } x = \frac{1}{2}, \quad \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = 1 + \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{3\left(\frac{1}{2}\right)^4}{8} + \dots$$

$$\frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = 1 + \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{3\left(\frac{1}{2}\right)^4}{8} + \dots$$

$$\Rightarrow \sqrt{3} = \frac{256}{147}$$

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(a)

$$\text{Area} = \int_0^{1/4} x^2 \sqrt{1-4x^2} \, dx$$

$$\text{Let } x = \frac{1}{2} \sin \theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{2} \cos \theta$$

$$\text{When } x = 0, \theta = 0; \quad x = \frac{1}{4}, \theta = \frac{\pi}{6}.$$

Substituting,

$$\text{Area} = \int_0^{\pi/6} \frac{1}{4} \sin^2 \theta \sqrt{1-\sin^2 \theta} \cdot \frac{1}{2} \cos \theta \, d\theta$$

$$= \frac{1}{8} \int_0^{\pi/6} \sin^2 \theta \cos^2 \theta \, d\theta$$

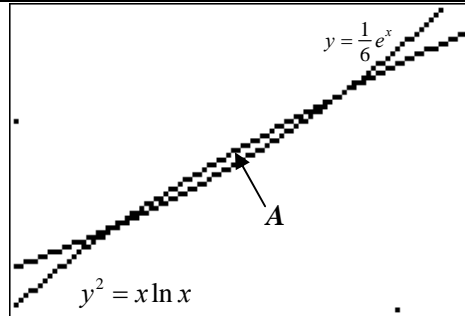
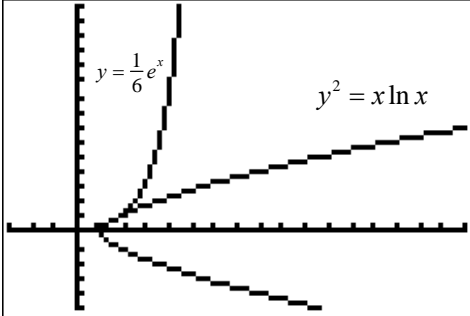
$$= \frac{1}{32} \int_0^{\pi/6} \sin^2 2\theta \, d\theta$$

$$= \frac{1}{64} \int_0^{\pi/6} 1 - \cos 4\theta \, d\theta$$

$$= \frac{1}{64} \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/6}$$

$$= \frac{1}{64} \left[\frac{\pi}{6} - \frac{\sqrt{3}}{8} \right]$$

10
(b)



From G.C., approximate coordinates of the points of intersection of the two curves are $(1.36759, 0.65435)$ and $(1.87156, 1.08307)$.

$$\text{Volume} = \pi \int_{1.36759}^{1.87156} x \ln x - \left(\frac{1}{6} e^x \right)^2 \, dx = 0.0766 \text{ (3 s.f.)}$$

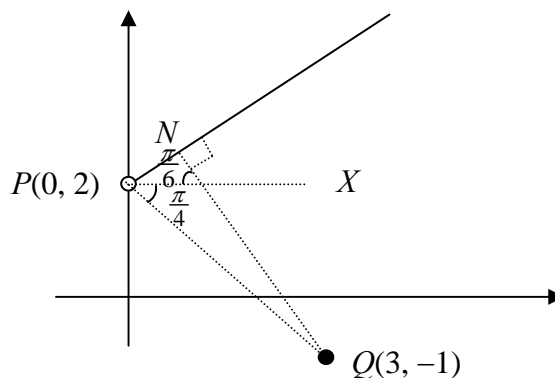
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$$\arg(iz + 2) = \frac{2}{3} \pi$$

$$\arg[i(z - 2i)] = \frac{2}{3} \pi$$

$$\arg i + \arg(z - 2i) = \frac{2}{3} \pi$$

$$\arg(z - 2i) = \frac{2}{3} \pi - \frac{\pi}{6} = \frac{\pi}{6}$$



$$\widehat{NPQ} = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$$

$$PQ = \sqrt{3^2 + 3^2} = 3\sqrt{2}$$

$$\sin \frac{5\pi}{12} = \frac{NQ}{3\sqrt{2}} \Rightarrow NQ = 4.10 \text{ (3 sig fig)}$$

$$\therefore m = 4.10, \quad m \geq 3\sqrt{2}$$

12 Asymptotes $y = x + a$, $x = 2$

(i)

$$\frac{dy}{dx} = 1 - \frac{4a^2}{(x-2)^2}$$

(ii)

For stationary points, $\frac{dy}{dx} = 0$

$$0 = 1 - \frac{4a^2}{(x-2)^2}$$

$$x = 2 \pm 2a$$

when $x = 2 + 2a$, $y = 2 + 5a$

when $x = 2 - 2a$, $y = 2 - 3a$

(iii)



13

$$w^3 = 1 \Rightarrow w = e^{\frac{2k\pi i}{3}}, k = -1, 0, 1$$

$$(z+i)^3 + (z-i)^3 = 0$$

$$(z+i)^3 = -(z-i)^3 = (-1)^3(z-i)^3$$

$$\left(\frac{z+i}{-z+i}\right)^3 = 1$$

$$\frac{z+i}{-z+i} = e^{\frac{2k\pi i}{3}}$$

$$z = \frac{i(-1 + e^{\frac{2k\pi i}{3}})}{1 + e^{\frac{2k\pi i}{3}}}$$

$$= \frac{i(-1 + \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3})}{1 + \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}}$$

$$= \frac{i \left[-1 + \left(1 - 2 \sin^2 \frac{k\pi}{3}\right) + i \left(2 \sin \frac{k\pi}{3} \cos \frac{k\pi}{3}\right) \right]}{1 + \left(2 \cos \frac{k\pi}{3} - 1\right) + i \left(2 \sin \frac{k\pi}{3} \cos \frac{k\pi}{3}\right)}$$

$$= \frac{-2 \sin \frac{k\pi}{3} \left(\cos \frac{k\pi}{3} + i \sin \frac{k\pi}{3} \right)}{2 \cos \frac{k\pi}{3} \left(\cos \frac{k\pi}{3} + i \sin \frac{k\pi}{3} \right)}$$

$$= -\tan \frac{k\pi}{3}$$

ALTERNATIVE:

$$\begin{aligned}
 z &= \frac{i(-1 + e^{\frac{2k\pi}{3}i})}{1 + e^{\frac{2k\pi}{3}i}} \\
 &= \frac{i(-1 + e^{\frac{2k\pi}{3}i})}{1 + e^{\frac{2k\pi}{3}i}} \cdot \frac{e^{-\frac{k\pi}{3}i}}{e^{-\frac{k\pi}{3}i}} \\
 &= \frac{i(-e^{-\frac{k\pi}{3}i} + e^{\frac{k\pi}{3}i})}{e^{-\frac{k\pi}{3}i} + e^{\frac{k\pi}{3}i}} \\
 &= \frac{i(2i \sin \frac{k\pi}{3})}{2 \cos \frac{k\pi}{3}} \\
 &= -\tan \frac{k\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 (z+i)^3 + (z-i)^3 &= 0 \\
 (z^3 + 3z^2i - 3z - i) + (z^3 - 3z^2i - 3zi^2 + i) &= 0
 \end{aligned}$$

$$2(z^3 - 3z) = 0$$

$$z(z^2 - 3) = 0$$

$$z = 0, \pm\sqrt{3}$$

Since $z = -\tan \frac{\pi}{3}$ is negative $\therefore \tan \frac{\pi}{3} = \sqrt{3}$.

14(i)

$$\angle BOC = \cos^{-1} \frac{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}}{\sqrt{11}\sqrt{9}} = 107.5^\circ \text{ (1 d.p)}$$

$$\begin{aligned}
 \text{(ii) Area of } \triangle OBC &= \frac{1}{2} |\overline{OB}| |\overline{OC}| \sin(107.5^\circ) \\
 &= \frac{1}{2} \sqrt{11} \sqrt{81} \sin(107.5^\circ) \\
 &= 14.2 \text{ units}^2
 \end{aligned}$$

$$\text{(iii) } \overline{CP} = \begin{pmatrix} 2-2t \\ 1+4t \\ 3-4t \end{pmatrix} - \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix} = \begin{pmatrix} -1-2t \\ 7+4t \\ -3-4t \end{pmatrix}, \quad \overline{CB} = \begin{pmatrix} -2 \\ 9 \\ -5 \end{pmatrix}$$

Since,

$$\left| \frac{\overline{CP} \cdot \overline{CB}}{|\overline{CB}|} \right| = \sqrt{110}$$

$$\left| \frac{\begin{pmatrix} -1-2t \\ 7+4t \\ -3-4t \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 9 \\ -5 \end{pmatrix}}{\sqrt{110}} \right| = \sqrt{110}$$

$$60t + 80 = 110 \quad \text{or} \quad 60t + 80 = -110$$

$$t = \frac{1}{2} \quad \text{or} \quad -3\frac{1}{6} \quad (-3.17)$$

(iv)

$$\overrightarrow{PA} = -2t \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \quad \overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

Since $\overrightarrow{PA} = -2t(\overrightarrow{AB})$, hence, P , A and B are collinear for all values of t .

(v)

$$[\text{OR } \overrightarrow{PB} = (1-2t)(\overrightarrow{AB}), \text{ OR } \overrightarrow{PB} = \left(\frac{-1}{2t} + 1\right)(\overrightarrow{PA})]$$

$\triangle OBC$ and $\triangle OPC$ share the same base of OC . Since $\overline{OC} \parallel \overline{BP}$, $\triangle OBC$ and $\triangle OPC$ also has the same perpendicular height.

\therefore Area of $\triangle OPC =$ Area of $\triangle OBC = 14.2 \text{ units}^2$.